Analytic Number Theory Notes

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The aim of this course is to introduce the standard techniques in analytic number theory through the study of two classical results, namely, the prime number theorem and Dirichlet's theorem on primes in arithmetic progressions. Major topics: Arithmetical functions. Mertens' estimates. Riemann zeta function. Prime number theorem. Characters of abelian groups. Dirichlet's theorem on primes in arithmetic progressions. Textbooks: [Apo98] and [Cha12].

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Contents

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1. Introduction and elementary results

1.1. Big Oh notation

Definition 1.1

Let g and h be real-valued functions. We say $g(x) = O(h(x))$ if there exists a $C > 0$ such that $|g(x)| \leq Ch(x)$ for all x. *Vinogradov notation* is writing $h \ll g$ if $h = O(g)$.

One useful fact is that $f_i(x) = g_i(x) + O(h_i(x))$ for $i = 1, 2$ implies that $f_1(x) + f_2(x) = g_1(x) + g_2(x) + O(h_1(x) + h_2(x))$.

Definition 1.2

Given a function $h(x)$, we say $g(x) = o(h(x))$ if $\lim_{x \to a} g(x)/h(x) = 0$. If a is not mentioned, then we assume $a = \infty$.

Definition 1.3

If f, g are positive functions, then $f \sim g$ as $x \to a$ means $\lim_{x \to a} f(x)/g(x) = 1$. Again, if a is not mentioned, assume $a = \infty$.

Definition 1.4

If f, g are positive functions, then $f \asymp g$ for $x \in S$ means $f = O(g)$ and $g = O(f)$ for $x \in S$.

1.2. Arithmetic functions

Definition 1.5

A function f is **arithmetic** if it is a function from $\mathbb{Z}_{>0}$ to $\mathbb{C}$. An arithmetic function $f \colon \mathbb{Z}_{>0} \to \mathbb{C}$ is **multiplicative** if $f(1) = 1$ and $f(ab) = f(a)f(b)$ when a and b are coprime.

Example 1.1 – Euler's totient function $\phi(n) := \#\{1 \leq m \leq n : (n, m) = 1\}$ is multiplicative.

1.3. Counting lattice points on the circle I

The rest of this section is dedicated to covering results that can be proven in elementary ways. For this course, it basically means we don't use complex analysis, or that a high-schooler with calculus knowledge could follow these proofs (though we might occasionally appeal to some algebra results).

Consider the function $r(n) := \#\left\{(a, b) \in \mathbb{Z}^2 : n = a^2 + b^2\right\}$ of Pythagorean triples with hypotenuse n. This function is *not* multiplicative: $r(1) = 4$.

If p is a prime equivalent to 3 modulo 4, then $r(p) = 0$. Since there are infinitely many such primes, $\liminf_{n\to\infty} r(n) = 0$.

We can also try to approximate the function $R(N) := \sum_{n=0}^{N} r(n)$. This counts the number of lattice points in the ball of radius $\sqrt{N}$. Suppose for each lattice point inside the ball, we place a box whose bottom-left corner is that lattice point. The number of lattice points is precisely the area of the (disjoint) union of these squares, S. On the other hand, S is contained in the ball of radius $\sqrt{N} + \sqrt{2}$ centered at the origin by the triangle inequality. Moreover, S contains the ball of radius $\sqrt{N} - \sqrt{2}$. Therefore, $\pi(\sqrt{N} - \sqrt{2})^2 \le R(N) \le \pi(\sqrt{N} + \sqrt{2})^2$, which implies $R(N) = \pi N + O(\sqrt{N})$.

1.4. Asymptotics of the divisor function I

Let $d(n) := \#\{d : d \mid n\}$ be the **divisor function**. It is multiplicative. More explicitly, if $n = p_1^{e_1} \cdots p_k^{e_k}$, then $d(n) = \prod_i (e_i + 1)$. Because there are infinitely many primes, $\liminf_n d(n) = 2$.

Proposition 1.1

For all $\Delta > 0$, there is a sequence (n_i) such that $d(n_i)/(\log n_i)^{\Delta} \to \infty$ as $i \to \infty$.

Proof. Let $k \le \Delta < k+1$, and let p_i be the ith prime. For $n_i := (p_1 \cdots p_{k+1})^i$, $d(n_i) = (i+1)^{k+1} > i^{k+1}$. But

$$i^{k+1} = \left(\frac{\log n_i}{\log p_1 \cdots p_{k+1}}\right)^{k+1} = \underbrace{(\log p_1 \cdots p_{k+1})^{-(k+1)}}_{=:c} (\log n_i)^{k+1},$$

where we note that c is independent of i. Write $k+1 = \Delta + \delta$ for $\delta \in (0,1]$. Then

$$d(n_i)/\log(n_i)^{\Delta} > c\log(n_i)^{k+1-\Delta} = c\log(n_i)^{\delta} \xrightarrow{i\to\infty} \infty. \qquad \square$$

1.5. Asymptotics of the divisor function II

January 16, 2026 We'll use the following lemma to get an upper (little-oh) bound on $d(n)$.

Theorem 1.2 (Prime power convergence implies convergence)

If $f\colon \mathbb{Z}_{>0} \to \mathbb{C}$ is multiplicative and $f(p^n) \to 0$ as $p^n \to \infty$, then $f(n) \to 0$ as $n \to \infty$.

Remark 1.3. This is not the same as $f(p^n) \to 0$ as $n \to \infty$ for all n. For example, f could be the function defined by $f(p^n) = 2^{-n}$ for each prime p and extended by multiplicativity. Then $f(p^n) \to 0$ as $n \to \infty$ for all primes p, but $f(p) = \frac{1}{2}$ for every prime, so f never converges to 0 because there are infinitely many primes.

We can rephrase the theorem as follows: let P be the set of prime powers. Then $f|_P(n) \to 0$ implies $f(n) \to 0$.

Proof of Theorem 1.2. For a given M, the set of integers where each prime power factor is less than M is finite. Let $P(M)$ be the maximal such number. For $n > P(M)$, we know $n = p^a \cdot n'$ for some $p^a > M$.

We claim f is bounded. There exists a $B > 0$ such that for all $p^n > B$, $|f(p^n)| < 1$. There are only finitely many $p^n \le B$, so there is an $A > 0$ such that $|f(p^n)| < A$ for $p^n \le B$. Let $C := \#\{p^n < B\}$. Then $|f(p_1^{e_1} \cdots p_k^{e_k})| = |f(n)| < A^C$ for all n.

Let $\varepsilon > 0$ and N be such that $p^n > N$ implies $|f(p^n)| < \frac{\varepsilon}{A^C}$. For $n > P(N)$, we can write $n = p^a \cdot n'$ for some $p^a > N$. By multiplicativity,

$$|f(n)| = |f(p^a)| \cdot |f(n')| < \frac{\varepsilon}{A^C} \cdot A^C = \varepsilon,$$

which implies $f(n) \to 0$. □

Theorem 1.4 (The divisor function grows slower than any power of n)
$d(n) = o(n^\delta)$ for all $\delta > 0$.

Proof. Let $f(n) = d(n)/n^\delta$. We note that f is multiplicative as the quotient of multiplicative functions. Then

$$\begin{aligned} f(p^n) &= \frac{d(p^n)}{p^{\delta n}} \\ &= \frac{n+1}{p^{\delta n}} \\ &\le \frac{2n}{p^{\delta n}} \\ &= \frac{2\log p^n/\log p}{p^{\delta n}} \\ &= \frac{2}{\log p} \cdot \frac{\log p^n}{p^{\delta n}} \\ &< \frac{2}{\log 2} \cdot \frac{\log N}{N^\delta}. && (N = p^n) \end{aligned}$$

This tends to 0 as $N \to \infty$, so Theorem 1.2 implies that $f(n) \to 0$. □

1.6. Counting lattice points on the circle II

The function $r(n)$ of the number of (x, y) such that $a^2 + b^2 = n$, was defined before, and was not multiplicative. Consider first the function $r_0(n) := \#\left\{(a,b) \in \mathbb{Z}^2 : n = a^2 + b^2, (a,b) = 1\right\}$ of the number of *primitive* solutions to $a^2 + b^2 = n$. We have

$$r(n) = \sum_{\substack{d>0 \\ d^2 | n}} r_0(n/d^2).$$

We consider only the solutions in the "fundamental domain" under $\frac{\pi}{2}$ rotation: $r_0'(n) := \left\{(a,b) \in \mathbb{Z}^2 : n = a^2 + b^2, a \ge 0, b > 0, (a,b) = 1\right\}$. Then $r_0(n) = 4r_0'(n)$.

We claim r_0' is multiplicative. This is by making use of the bijection

$$\left\{(a,b) \in \mathbb{Z}^2 : n = a^2+b^2, a \geq 0, b > 0, (a,b) = 1\right\} \longleftrightarrow \left\{x \in \mathbb{Z}/n : x^2 \equiv -1 \pmod{n}\right\}$$
$$(a,b) \mapsto a/b.$$

From the theory of binary quadratic forms,

$$r(n) = 4\left(\#\{d \in \mathbb{Z}_{>0} : d \mid n, d \equiv 1 \pmod{4}\} - \#\{d \in \mathbb{Z}_{>0} : d \mid n, d \equiv 3 \pmod{4}\}\right).$$

Therefore, $r(n) \leq 4d(n) = o(n^\delta)$ for any $\delta > 0$, so $r(n) = o(n^\delta)$.

Theorem 1.5
If $g\colon [1,\infty) \to \mathbb{R}_{>0}$ is monotone decreasing, then

$$\sum_{1 \leq n \leq x} g(n) = \int_1^x g(t)\,dt + A + O(g(x))$$

for some constant A depending on g.

Proof. Since g is decreasing,

$$g(n) \geq \int_n^{n+1} g(t)\,dt \geq g(n+1),$$

so

$$0 \leq A_n \coloneqq g(n) - \int_n^{n+1} g(t)\,dt \leq g(n) - g(n+1).$$

For some $N < M$,

$$\sum_{n=N}^{M} A_n \leq g(N) - g(M+1) \leq g(N).$$

Set

$$A \coloneqq \sum_{n=1}^{\infty} A_n \leq g(1).$$

Then

$$\begin{aligned} A &= \sum_{n=1}^{N} A_n + \sum_{n=N+1}^{\infty} A_n \\ &\leq \sum_{n=1}^{N} g(n) - \int_n^{n+1} g(t)\,dt + g(N+1) \\ &= \sum_{n=1}^{N} g(n) - \int_1^{N+1} g(t)\,dt + g(N+1). \end{aligned}$$

Setting $N = \lfloor x \rfloor$, we have

$$\begin{aligned}
\sum_{1 \le n \le x} g(n) &= A + \int_1^{\lfloor x \rfloor + 1} g(t)\,dt + O(g(\lfloor x \rfloor + 1)) \\
&\le A + \int_1^{\lfloor x \rfloor + 1} g(t)\,dt + O(g(x)) \\
&= A + \int_1^{x} g(t)\,dt + \underbrace{\int_x^{\lfloor x \rfloor + 1} g(t)\,dt}_{\le g(x)} + O(g(x)) \\
&= A + \int_1^{x} g(t)\,dt + O(g(x)).
\end{aligned}$$

□

Corollary 1.6

There exists a constant γ (*Euler's constant*) such that

$$\sum_{1 \le n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).$$

Corollary 1.7

There exists a constant B such that

$$\sum_{1 \le n \le x} \frac{1}{n \log n} = \log \log x + B + O\left(\frac{1}{x \log x}\right).$$

1.7. Asymptotics of the average of the divisor function

Let $D(N) := \sum_{n=1}^{N} d(n)$. Another way to count this as the number of lattice points $(x, y) \in \mathbb{Z}^2_{>0}$ such that $xy \le N$. This is also the number of lattice points below the graph N/x (from $x = 1$ to $x = N$).

Theorem 1.8

$D(N) = N \log N + O(N)$.

Proof. We have

$$D(N) = \sum_{n=1}^{N} \left\lfloor \frac{N}{n} \right\rfloor.$$

Set $\theta_x := N/x - \lfloor N/x \rfloor$. Then

$$\begin{aligned} D(N) &= \sum_{n=1}^{N} \frac{N}{n} - \underbrace{\theta_n}_{\in[0,1)} \\ &= N \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \theta_n \\ &= N \log n + N\gamma - \sum_{n=1}^{N} \theta_n + NO\left(\frac{1}{N}\right) \\ &= N \log N + O(N). \end{aligned}$$

□

We'll now remove the $O(N)$ error. We want to reduce the number of θ_n's that we have to sum over, since they contribute to the error.

Theorem 1.9 (Dirichlet)
$D(N) = N \log N + (2\gamma - 1)N + O(\sqrt{N})$.

Proof (sketch). The lattice points are symmetric about the axis $y = x$. Note that $y = x$ intersects $y = N/x$ at $(\sqrt{N}, \sqrt{N})$. If A is the number of lattice points between $x = 1$ and $x = \sqrt{N}$, then symmetry gives us

$$D(N) = 2A - \left\lfloor \sqrt{N} \right\rfloor^2$$

(see Figure 1 for a visualization). We have that

$$A = \sum_{1 \le n \le \sqrt{N}} \left\lfloor \frac{N}{n} \right\rfloor = N\left(\log \sqrt{N} + \gamma + O\left(\frac{1}{\sqrt{N}}\right)\right) + O(\sqrt{N}),$$

and

$$\left\lfloor \sqrt{N} \right\rfloor^2 = (\sqrt{N} - \theta_{\sqrt{N}})^2 = N + O(\sqrt{N}).$$

Combining these gives us the result. □

Remark 1.10.

- $O(\sqrt{N})$ was improved to $O(N^{1/3} \log N)$ by Voronoi.
- **Conjecture:** the error can be further improved to $O(N^{1/4+\varepsilon})$ for $\varepsilon > 0$, but not $\varepsilon = 0$.

1.8. The σ function

Define $\sigma_k(n) := \sum_{d|n} d^k$. For example, $\sigma_0(n) = d(n)$.

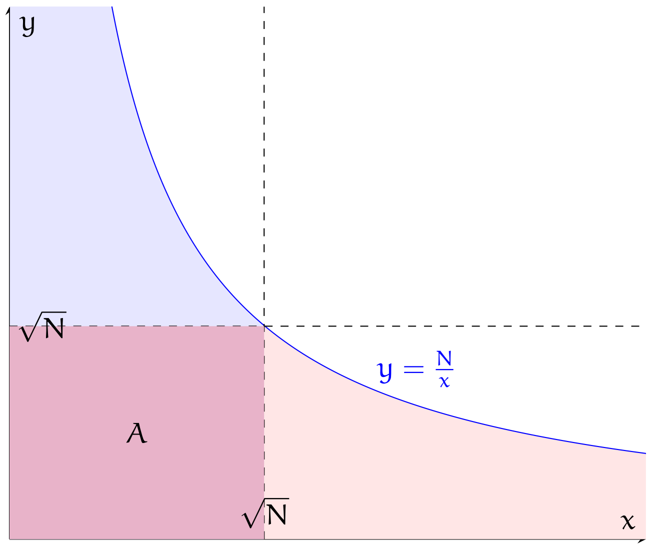

Figure 1: Visualization of the proof of Theorem 1.9.

Lemma 1.11

σ_k is multiplicative for all k.

Proof. If $(n, m) = 1$, then there is a bijection $\{d_1 \mid m\} \times \{d_2 \mid n\} \longleftrightarrow \{d \mid mn\}$ given by $(d_1, d_2) \mapsto d_1 d_2$. After expanding $\sigma_k(m)\sigma_k(n)$, the result follows. $\square$

Lemma 1.12

If $n > 1$ is written as $n = \prod_{i=1}^{r} p_i^{e_i}$, then

$$\sigma_k(n) = \prod_{i=1}^{r} \frac{p_i^{(e_i+1)k} - 1}{p_i^k - 1}.$$

Proof. First find a formula for p^e, then use geometric series + multiplicativity. $\square$

1.9. Perfect numbers

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Definition 1.6

A number n is **perfect** if $\sigma_1(n) = 2n$.

Example 1.2 –

- $\sigma_1(6) = \sigma_1(2 \cdot 3) = 1 + 2 + 3 + 6 = 12$.

- $\sigma_1(28) = \sigma_1(2^2 \cdot 7) = 1 + 2 + 4 + 7 + 14 + 28 = 56.$

Definition 1.7

A **Mersenne number** is an integer of the form $2^n - 1$. If it is prime, we call it a **Mersenne prime**.

Proposition 1.13

If $2^{n+1} - 1$ is prime, then $2^n(2^{n+1} - 1)$ is perfect.

Proof. We have

$$\begin{aligned}\sigma_1(2^n(2^{n+1} - 1)) &= \sigma_1(2^n)\sigma_1(2^{n+1} - 1) \\ &= (2^{n+1} - 1)(2^{n+1}).\end{aligned}$$

□

Theorem 1.14 (Euler)

All even perfect numbers are of the form in Proposition 1.13.

Proof. Suppose $N = 2^n N'$ is perfect and N' is odd. Since N is perfect,

$$\sigma_1(N) = 2N = 2^{n+1}N'.$$

On the other hand, by multiplicativity,

$$\sigma_1(N) = (2^{n+1} - 1)\sigma_1(N').$$

Therefore, $2^{n+1} - 1 \mid N'$, so set $N' = (2^{n+1} - 1)N''$. The goal now is to show $N'' = 1$. Note that

$$\sigma_1(N') = 2^{n+1} \cdot \frac{\sigma_1(N)}{2^{n+1} - 1} = 2^{n+1}N''.$$

We have that $N'' < N'$, but $N' + N'' = 2^{n+1}N'' = \sigma_1(N')$. But if N' and N'' divide N' and sum to $\sigma(N')$, then N' is prime, so $N'' = 1$. □

Remark 1.15. It is open whether there are infinitely many even perfect numbers. Moreover, it open whether there are *odd* perfect numbers.

1.10. Asymptotics for $n!$

Theorem 1.16

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

This follows from *Stirling's formula*:

$$\lim_{x \to \infty} \frac{\Gamma(x+1)}{(x/e)^x x^{1/2}} = \sqrt{2\pi},$$

but a crude estimate can be achieved by approximating the integral of $\log x$.

1.11. Multiplicative convolution

Definition 1.8

Let $\mathscr{A} := \{f \colon \mathbb{Z}_{>0} \to \mathbb{C}\}$. This forms a $\mathbb{C}$-algebra under pointwise addition and multiplication, with $1(n) = 1$ for all n. Consider also the functions $\delta(n)$, which is 1 if $n = 1$ and 0 otherwise, and $\mathrm{id}(n) = n$ for all n.

Definition 1.9

The **convolution** operation $- * -\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is defined by

$$(f * g)(n) := \sum_{\substack{d>0 \\ d \mid n}} f(d) g\left(\frac{n}{d}\right).$$

We call $f * g$ the **multiplicative convolution** of f and g.

Proposition 1.17

$*$ on $\mathscr{A}$ is commutative, associative, distributive over $+$, and has δ as the identity.

Proposition 1.18

If f and g are multiplicative, then so is $f * g$.

Definition 1.10

The **Möbius function** is an arithmetic function $\mu\colon \mathbb{Z}_{>0} \to \mathbb{C}$ defined as follows:

1. $\mu(1) = 1$,
2. $\mu(n) = (-1)^k$ if n is the product of k distinct primes,
3. $\mu(n) = 0$ otherwise.

Proposition 1.19
μ is multiplicative and

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mu * 1 = \delta$.

Proof. Let $n = p_1^{a_1} \cdots p_k^{a_k}$ for distinct p_i. Then the only times $\mu(d)$ is nonzero is for $p_{i_1} \cdots p_{i_j}$ for $1 \le j \le k$ and $i_1, \dots, i_j$ distinct. There are precisely $\binom{k}{j}$ such divisors of n for each j. Then

$$\sum_{d|n} \mu(d) = 1 - \underbrace{\binom{k}{1}}_{p_1, p_2, \dots, p_k} + \underbrace{\binom{k}{2}}_{p_1p_2, p_1p_3, \dots, p_{n-1}p_n} - \cdots + (-1)^k \underbrace{\binom{k}{k}}_{p_1 \cdots p_k} = (1-1)^k = 0. \qquad \square$$

The next theorem demonstrates why we are looking at the Möbius function:

Theorem 1.20 (First Möbius inversion formula)
If f and g are arithmetic functions, then $g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(d)g(n/d)$.

1.12. The Van Mangoldt function

Definition 1.11
The **van Mangoldt function** Λ is an arithmetic function defined by

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^m \text{ for some } m > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Proposition 1.21
$\sum_{d|n} \Lambda(d) = \log n$.

Proof. If $n = p_1^{e_1} \cdots p_n^{e_n}$, then

$$\sum_{d|n} \Lambda(d) = \sum_i \sum_{j \le e_i} \Lambda(p_i^j) = \sum_i e_i \log p_i = \log p_1^{e_1} \cdots p_n^{e_n} = \log n. \qquad \square$$

By Möbius inversion, we may write

$$\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d) = \sum_{d|n} \mu(d)(\log n - \log d) = -\sum_{d|n} \mu(d) \log d.$$

Remark 1.22. Let $\psi(x) \coloneqq \sum_{n \le x} \Lambda(n)$. Conjecturally,

$$\psi(x) = x + O(x^{1/2}(\log x)^2).$$

The exponent in the big-oh term being $1/2$ is equivalent to the Riemann hypothesis.

Theorem 1.23 (Second Möbius inversion formula)
If $f, g\colon \mathbb{R}_{\ge 1} \to \mathbb{C}$ are functions, then $g(x) = \sum_{n \le x} f(x/n) \iff f(x) = \sum_{n \le x} \mu(n) g(x/n)$.

1.13. Asymptotics of the Euler function

Recall that $\phi(n) = \#\{1 \le m \le n : (n, m) = 1\} = |(\mathbb{Z}/n)^\times|$.

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Proposition 1.24
$\sum_{d|n} \phi(d) = n$. In other words, $\phi * 1 = \mathrm{id}$.

Proof. On one hand $|\mathbb{Z}/n| = n$. On the other hand, any $a \in \mathbb{Z}/n$ generates a cyclic subgroup. Moreover, a cyclic subgroup of order d has $\phi(d)$ generators. Therefore, since the subgroups of $\mathbb{Z}/n$ are $d\mathbb{Z}/n$ for each divisor $d \mid n$,

$$n = |\mathbb{Z}/n| = \sum_{\substack{G \le \mathbb{Z}/n \\ G \text{ cyclic}}} \#\{\text{generators of } G\} = \sum_{d|n} \phi(d). \qquad \square$$

By the first Möbius inversion formula (1.20),

$$\phi(n) = \sum_{d|n} \mu(d) \cdot \frac{n}{d} = n \sum_{d|n} \frac{\mu(d)}{d}.$$

Since ϕ is multiplicative, and $\phi(p^a) = p^a \left(1 - \frac{1}{p}\right)$, for $n = p_1^{a_1} \cdots p_k^{a_k}$, we have

$$\begin{aligned} \phi(n) &= n \prod_{i=1}^{n} \left(1 - \frac{1}{p_i}\right) \\ &= n \prod_{p|n} \left(1 - \frac{1}{p}\right). \end{aligned}$$

We'll now study the asymptotics of $\phi(n)$. We know that $\phi(n) < n$, and $\limsup_{n\to\infty} \phi(n)/n = 1$ because there are arbitrarily large primes.

Proposition 1.25
$\lim_{n\to\infty} \phi(n)/n^{1-\delta} = \infty$ for $\delta > 0$.

Proof. Assume $\delta \in (0, 1]$ (otherwise, the result is obvious). Consider the multiplicative function $f(n) := n^{1-\delta}/\phi(n)$. If $n = p^a$ is a prime power, then

$$f(n) = \frac{n^{1-\delta}}{n\left(1-\frac{1}{p}\right)} = \frac{1}{n^\delta\left(1-\frac{1}{p}\right)}$$
$$\leq \frac{1}{\frac{1}{2}n^\delta} \xrightarrow{n\to\infty} 0.$$

By Theorem 1.2, $f(n) \xrightarrow{n\to\infty} 0$. Taking the reciprocal gives the result. □

In particular, if $\Delta < 1$, then $\phi(n) \leq Cn^\Delta$ implies $\phi(n)/n^\Delta$ is bounded, which contradicts Proposition 1.25. Therefore, $\phi(n) \neq O(n^\Delta)$ for any $\Delta < 1$.

1.14. Asymptotics of the average of the Euler function

We now consider the average size of $\phi(n)$. Define $\Phi(x) := \sum_{n\leq x} \phi(n)$.

Theorem 1.26 (Mertens)
$\Phi(t) = \frac{3t^2}{\pi^2} + O(t \log t)$.

Proof. We can write

$$\Phi(t) = \sum_{1\leq n\leq t} \phi(n) = \sum_{1\leq n\leq t} \sum_{\substack{1\leq m\leq n\\(n,m)=1}} 1$$
$$= \#\left\{(n, m) \in \mathbb{Z}^2_{>0} : 1 \leq m \leq n \leq t, (m, n) = 1\right\}.$$

In other words, we want to count the number of lattice points $(0, t]^2$ with coprime coordinates and y-coordinate $\geq$ x-coordinate. Using the obvious symmetry across the line $y = x$, we instead consider $\Psi(t)$, the number of lattice points in $(0, t]^2$, with the relation

$$\Psi(t) = 2\Phi(t) - 1,$$

where we subtract 1 because we double count $(1, 1)$.

Note that $(m, n) = d \iff d \mid m, d \mid n$, and $(\frac{m}{d}, \frac{n}{d}) = 1$. Therefore, we have a bijection

$$\left\{(m, n) \in \mathbb{Z}^2_{>0} : m, n \leq t, (m, n) = d\right\} \longleftrightarrow \left\{(m, n) \in \mathbb{Z}^2_{>0} : m, n \leq \frac{t}{d}, (m, n) = 1\right\}.$$

Now consider all $\lfloor t\rfloor^2$ lattice points in $(0, t]^2$:

$$\lfloor t\rfloor^2 = \sum_{1\leq d\leq t} \sum_{\substack{m,n\leq t\\(m,n)=d}} 1 = \sum_{1\leq d\leq t} \Psi\left(\frac{t}{d}\right).$$

Let $\theta_d = \frac{t}{d} - \lfloor \frac{t}{d} \rfloor$ be our error term, which lies in $[0, 1)$. By the second Möbius inversion formula,

$$\begin{aligned}\Psi(t) &= \sum_{d \leq t} \mu(d) \left(\frac{t}{d} - \theta_d \right)^2 \\ &= t^2 \sum_{d \leq t} \frac{\mu(d)}{d^2} - 2t \sum_{d \leq t} \frac{\mu(d)}{d} \theta_d + \sum_{d \leq t} \mu(d) \theta_d^2.\end{aligned}$$

We first bound the second and third terms, which we treat as errors:

$$\begin{aligned}\left| 2t \sum_{d \leq t} \frac{\mu(d)}{d} \theta_d \right| &\leq 2t \sum_{d \leq t} \frac{1}{d} \\ &= 2t \left(\log t + \gamma + O\left(\frac{1}{t} \right) \right) \\ &= O(t \log t) \\ \left| \sum_{d \leq t} \theta_d^2 \mu(d) \right| &\leq \sum_{d \leq t} 1 \\ &= O(t).\end{aligned}$$

So both terms absorb into $O(t \log t)$.

For the main term, we pass to the infinite sum. To do this, we show that the error of the remaining terms grows slowly ($O\left(\frac{1}{t}\right)$):

$$\sum_{1 \leq d \leq t} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d > t} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(\frac{1}{t} \right).$$

This is done by comparing to the integral. We have that

$$\left(\sum_{d=1}^{\infty} \frac{1}{d^2} \right) \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \right) = \sum_{n=1}^{\infty} \frac{(1 * \mu)(n)}{n^2},$$

which can be seen because the series on the LHS absolutely converge, so we can rearrange, yielding

$$\begin{aligned}\left(\sum_{d=1}^{\infty} \frac{1}{d^2} \right) \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \right) &= \sum_{n} \sum_{\substack{d_1, d_2 \\ d_1 d_2 = n}} \frac{\mu(d_2)}{(d_1 d_2)^2} \\ &= \sum_{n} \frac{1}{n^2} \sum_{\substack{d_1, d_2 \\ d_1 d_2 = n}} \mu(d_2) \\ &= \sum_{n} \frac{1}{n^2} (1 * \mu)(n).\end{aligned}$$

Since $1 * \mu = \delta$, the RHS is equal to 1. Finally, we note that $\sum_{d=1}^{\infty} \frac{1}{d^2} = \zeta(2) = \frac{\pi^2}{6}$, so rearranging gives us

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2}.$$

Plugging this back gives us the result. $\square$

Theorem 1.26 implies that

$$\begin{aligned} &\mathbb{P}\left(\text{a lattice point in } (0,t]^2 \text{ has coprime coordinates}\right) \\ &= \Psi(t) / \left\lfloor t^2 \right\rfloor \xrightarrow{t\to\infty} \frac{6}{\pi^2} \approx 0.608. \end{aligned}$$

1.15. The relationship between ϕ and σ

Let $\sigma = \sigma_1$.

Theorem 1.27

There exists a constant $C > 0$ such that

$$C < \frac{\sigma(n)\phi(n)}{n^2} < 1$$

for all $n \geq 2$.

Proof. Let $n = p_1^{a_1} \cdots p_k^{a_k}$. Recall that

$$\begin{aligned} \sigma(n) &= n \prod_{p|n} \frac{1 - p^{-a-1}}{1 - p^{-1}} \\ \phi(n) &= n \prod_{p|n} 1 - \frac{1}{p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\sigma(n)\phi(n)}{n^2} &= \prod_{p|n} 1 - \frac{1}{p^{a+1}} \\ &> \frac{1}{\prod_{p|n}\left(1 - \frac{1}{p^2}\right)^{-1}} \\ &= \frac{1}{\zeta(2)} \\ &= \frac{6}{\pi^2} =: C. \end{aligned}$$

$\square$

Remark 1.28 (Equivalent formulations of the Riemann hypothesis (RH)). If you don't know what the Riemann hypothesis is, come back to this after we state it. This is just to show that this problem is relevant to asymptotics we are currently studying.

- (Grönwall, 1913) Proved $\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}$.
- (Ramanujan, 1915) RH $\implies$ for all sufficiently large n, $\sigma(n) < e^{\gamma} n \log\log n$.
  - The largest known violation is $n = 5040$.
- (Robin, 1984) $\neg$RH $\implies$ there are infinitely many n such that $\sigma(n) \geq e^{\gamma} \log\log n$.
  - Moreover, RH $\implies$ $\sigma(n) < e^{\gamma} \log\log n$ holds for all $n > 5040$.
  - Therefore, RH is equivalent to $\sigma(n) < e^{\gamma} \log\log n$ for all $n > 5040$.
- (Nicholas, 1983) For $k \geq 1$, consider the *primorial* $N_k = p_1 \cdots p_k$.
  - RH $\implies$ for all $k \geq 1$, $\phi(N_k) < e^{-\gamma} N_k / \log\log N_k$.
  - $\neg$RH $\implies$ there are infinitely many k violating the above inequality.

1.16. Chebyshev's theorem on the distribution of primes

Definition 1.12

Let $\pi(x) := \#\{p \leq x : p \text{ is a prime}\}$ be the prime counting function.

Since there are infinitely many primes, we know that $\pi(x) \to \infty$ as $x \to \infty$. The famous *prime number theorem* says that $\frac{\pi(x)}{x/\log x} \to 1$ as $x \to \infty$.

1.16.1. A proof of the infinitude of primes

Theorem 1.29

$\sum_p \frac{1}{p}$ and $\prod_p \left(1 - \frac{1}{p}\right)^{-1}$ diverge. In particular, there are infinitely many primes.

Proof. Let

$$P(x) := \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1},$$
$$S(x) := \sum_{p \leq x} \frac{1}{p}.$$

If there are finitely many primes, then it is impossible for either of these functions to diverge. For $0 < u < 1$, the geometric series formula gives

$$(1-u)^{-1} = 1 + u + u^2 + \cdots > 1 + u + u^2 + \cdots + u^m.$$

Therefore, for some fixed m,

$$P(x) > \prod_{p \leq x} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^m}\right). \tag{1.2}$$

If $2^m > x$, then for each $n \leq x$, it can be factorized into (not necessarily unique) primes whose exponents are less than m. By expanding the product in (1.2) into a sum, we note that each term is positive, and $\frac{1}{n}$ is one such term. Therefore,

$$P(x) > \sum_{n \le x} \frac{1}{n}.$$

But as $x \to \infty$, the RHS diverges.

For $S(x)$, we use the Taylor expansion

$$\log\left(\frac{1}{1-u}\right) = u + \frac{u^2}{2} + \frac{u^3}{3} + \cdots,$$

which implies

$$\log\left(\frac{1}{1-u}\right) - u < \frac{1}{2}\left(u^2 + u^3 + \cdots\right) = \frac{u^2}{2(1-u)}.$$

Therefore,

$$\begin{aligned}
\log(P(x)) - S(x) &= \sum_{p \le x} \log\left(\frac{1}{1-p^{-1}}\right) - p^{-1} \\
&\le \sum_{p \le x} \frac{1}{2} \cdot \frac{1}{p(p-1)} \\
&< \sum_{n \le x} \frac{1}{2} \cdot \frac{1}{n(n-1)} \\
&= \frac{1}{2}. && \text{(telescoping)}
\end{aligned}$$

Combining this with the bound

$$P(x) > \sum_{n \le x} \frac{1}{n} > \int_1^{\lfloor x \rfloor + 1} \frac{1}{u}\, du > \log x,$$

we get that

$$S(x) > \log P(x) - \frac{1}{2} > \log\log x - \frac{1}{2},$$

which diverges as $x \to \infty$. □

1.16.2. Chebyshev's functions

January 28, 2026

Definition 1.13
Chebyshev's functions ϑ and ψ are defined for $x > 0$ as follows:

$$\vartheta(x) := \sum_{p \le x} \log p,$$
$$\psi(x) := \sum_{p^m \le x} \log p.$$

The sum in the definition of the ψ function is over all prime powers $\le x$.

Another way to write the ψ function is in terms of the van Mangoldt function (1.1):

$$\psi(x) = \sum_{n \le x} \Lambda(n).$$

In fact, we have that

$$e^{\vartheta(x)} = \prod_{p \le x} p, \qquad e^{\psi(x)} = \operatorname{lcm}(1, 2, \ldots, n = \lfloor x \rfloor).$$

Note that $p^m \le x \iff p \le x^{1/m}$, so

$$\psi(x) = \vartheta(x) + \vartheta(x^{1/2}) + \vartheta(x^{1/3}) + \cdots.$$

Moreover, we note that the highest power of p that could divide $\lfloor x \rfloor$ is $\left\lfloor \frac{\log x}{\log p} \right\rfloor$. Therefore, since $\log p$ appears once in the sum for $\psi(x)$ for each $1 \le k \le \left\lfloor \frac{\log x}{\log p} \right\rfloor$,

$$\psi(x) = \sum_{p \le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \cdot \log p \tag{1.3}$$

Lemma 1.30
Let

$$\ell_1 := \liminf_{x \to \infty} \frac{\pi(x)}{x/\log x}, \qquad L_1 := \limsup_{x \to \infty} \frac{\pi(x)}{x/\log x},$$
$$\ell_2 := \liminf_{x \to \infty} \frac{\vartheta(x)}{x}, \qquad L_2 := \limsup_{x \to \infty} \frac{\vartheta(x)}{x},$$
$$\ell_3 := \liminf_{x \to \infty} \frac{\psi(x)}{x}, \qquad L_3 := \limsup_{x \to \infty} \frac{\psi(x)}{x}.$$

Then $\ell_1 = \ell_2 = \ell_3$ and $L_1 = L_2 = L_3$.

Proof. It's clear that $\vartheta(x) \le \psi(x)$. Moreover,

$$\psi(x) \le \sum_{p \le x} \frac{\log x}{\log p} \cdot \log p = \log x \sum_{p \le x} 1 = \pi(x) \log x.$$

Therefore,
$$\vartheta(x) \le \psi(x) \le \pi(x) \log x.$$
Dividing by x and letting $x \to \infty$, we get
$$L_2 \le L_3 \le L_1.$$
We're now done if we show that $L_1 \le L_2$. Let $\alpha \in (0,1)$ and let $x > 1$. Then
$$\vartheta(x) \ge \sum_{x^\alpha < p \le x} \log p,$$
so
$$\vartheta(x) \ge \alpha(\log x)(\pi(x) - \pi(x^\alpha)).$$
Since $\pi(x^\alpha) < x^\alpha$, we have
$$\vartheta(x) > \alpha\pi(x)\log x - \alpha x^\alpha \log x,$$
which implies
$$\frac{\vartheta(x)}{x} > \alpha\pi(x)\frac{\log x}{x} - \alpha\frac{\log x}{x^{1-\alpha}}.$$
Since $\alpha \in (0,1)$, $(\log x / x^{1-\alpha}) \to 0$ as $x \to \infty$. Hence,
$$L_2 \ge \alpha L_1.$$
Taking $\alpha \to 1$ gives us $L_2 \ge L_1$, finishing $L_1 = L_2 = L_3$.

A similar proof follows for $\ell_1 = \ell_2 = \ell_3$. □

Theorem 1.31 (Chebyshev)
There exists constants $0 < a < A$ such that for sufficiently large x,
$$a < \frac{\pi(x)}{x/\log x} < A.$$
Hence, $\pi(x) \asymp \frac{x}{\log x}$.

We will prove $a < \log 2$ and $A > 4\log 2$. The proof will be elementary, only making use of Lemma 1.30, arithmetic, and basic bounds.

Proof of Theorem 1.31. We want to show that $\ell_1 \ge \log 2$ and $L_1 \le 4\log 2$. By Lemma 1.30, it suffices to show that (a) $\ell_3 \ge \log 2$ and (b) $L_2 \le 4\log 2$.

We consider the numbers
$$B_n := \binom{2n}{n}.$$
You might recognize these are the *Catalan numbers*: $\frac{1}{n+1}\binom{2n}{n}$ times $n+1$.

Lemma 1.32

For $n \in \mathbb{Z}_{>0}$,

$$\frac{1}{2n+1} 2^{2n} < B_n < 2^{2n}.$$

Proof (sketch). Expand $2^{2n} = (1+1)^{2n}$ with the binomial theorem. For the upper bound, just take the $\binom{2n}{n}$ term. For the lower bound, note that $\binom{2n}{k}$ is maximized when $k = n$. ■

We say that a prime p **divides n exactly k times** if $p^k \mid n$ but $p^{k+1} \nmid n$. We write $\nu_p(n) = k$.

For more information, see p-adic valuations.

Lemma 1.33 (How many 0's are at the end of $m!$?)

Let $m \geq 0$. Then

$$\nu_p(m!) = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \cdots .$$

Proof (idea). $m = 1 \cdot 2 \cdots m$. How many of $\{1, 2, \ldots, m\}$ are divisible by p exactly 1 time? 2 times? The sum of these numbers is what's written above. Now suppose p divides $k \in \{1, \ldots, m\}$ exactly a times. Show that k is counted a times in the sum. □

First, we show $L_2 \leq 4 \log 2$. First, note that if $n < p \leq 2n$, then $p \mid B_n$. From this, we get (using our bound on B_n in Lemma 1.32) that

$$\vartheta(2n) - \vartheta(n) = \sum_{n<p\leq 2n} \log p \leq \log(B_n) \overset{(1.32)}{<} \log(2^{2n}) = 2n \log 2.$$

Let $m = 2^a$. Then we can write $\vartheta(m)$ as a telescoping sum as follows (noting that $\vartheta(1) = 0$):

$$\begin{aligned} \vartheta(m) &= \sum_{k=0}^{a} \vartheta(2^k) - \vartheta(2^{k-1}) \\ &\leq \sum_{k=0}^{a} 2 \cdot 2^{k-1} \log 2 \\ &= (2^{a+1} - 1) \log 2. \end{aligned}$$

Now suppose $x \in [2^a, 2^{a+1})$. Then

$$\frac{\vartheta(x)}{x} \leq \frac{\vartheta(2^{a+1})}{2^a} \leq \left(\frac{2^{a+2} - 1}{2^a} \right) \log 2.$$

Taking $x \to \infty$ (and hence $a \to \infty$), we get our desired bound.

Now we show $\ell_3 \geq \log 2$. If n is an integer, then note that the prime factors of B_n are $\leq 2n$. Therefore,

$$\log(B_n) = \sum_{p \leq 2n} \nu_p(B_n) \log p = \sum_{p \leq 2n} (\nu_p((2n)!) - 2\nu_p(n!)) \log p,$$

where we are using the fact that $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ and $B_n = \frac{(2n)!}{(n!)(n!)}$. By Lemma 1.33,

$$\sum_{p \leq 2n} (\nu_p((2n)!) - 2\nu_p(n!)) \log p = \sum_{p \leq 2n} \sum_{k \geq 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) \log p.$$

Note that $\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \in \{0, 1\}$ (this follows from the more general fact that $\lfloor 2x \rfloor - 2 \lfloor x \rfloor \in \{0, 1\}$ for $x \geq 0$). In fact, when $p^k > 2n$, this value is zero. Therefore, we have a bound

$$\sum_{p \leq 2n} \sum_{k \geq 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) \log p \leq \sum_{p \leq 2n} \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \cdot \log p.$$

Noticing that the RHS is the formula for $\psi(2n)$ in (1.3), we have

$$\psi(2n) \geq \log(B_n) \overset{(1.32)}{>} \log\left(\frac{1}{2n+1} 2^{2n}\right) = 2n \log 2 - \log(2n+1).$$

Dividing through by $2n$, we get

$$\frac{\psi(2n)}{2n} > \log 2 - \frac{\log(2n+1)}{n}.$$

Now suppose $x \in [2n, 2n+2)$. Then

$$\frac{\psi(x)}{x} \geq \frac{\psi(2n)}{2n+2} > \left(\frac{2n+2}{2n}\right)\left(\log 2 - \frac{\log(2n+1)}{n}\right).$$

Taking $x \to \infty$ (and hence $n \to \infty$), we get our desired bound. $\square$

Remark 1.34 (The Erdős-Selberg dispute and an elementary proof of the prime number theorem)**.** If we prove $a = A = 1$ in Theorem 1.31, then we prove that $\pi(x) \sim \frac{x}{\log x}$, which is the prime number theorem. In the mid-1900s, this result was proved with elementary methods—a breakthrough because all previous proofs of the results used complex analysis. A dispute arose between Paul Erdős and Atle Selberg for who actually created the proof. See `https://www.math.columbia.edu/~goldfeld/ErdosSelbergDispute.pdf`.

1.17. Bertrand's postulate

Theorem 1.35 (Bertrand's postulate, proved by Chebyshev, 1852)
If n is a positive integer, then there is a prime p such that $n < p \leq 2n$.

We leave the proof for $n \le 64$ up to manual checking.

For all larger values of n, we will show that $\vartheta(2n) - \vartheta(n) > 0$ for all $n \in \mathbb{Z}_{>0}$. First, we prove the bound

$$\log(B_n) \le \vartheta(2n) - \vartheta(n) + \vartheta\left(\frac{2n}{3}\right) - \pi(\sqrt{2n})(\log 2 - \log 2n).$$

At the moment this expression seems random, but we'll motivate how to derive it in the proof.

Once we have this fact, we can rearrange terms to get

$$\vartheta(2n) - \vartheta(n) \ge \log(B_n) - \vartheta\left(\frac{2n}{3}\right) + \pi(\sqrt{2n})(\log 2 - \log 2n).$$

In order to bound the RHS below strictly by 0, we need a lower bound on $\log(B_n)$, and an upper bound on $\vartheta\left(\frac{2n}{3}\right)$ and $\pi(\sqrt{2n})(\log 2 - \log 2n)$ (since $\log 2 - \log 2n$ is non-positive). To do this we make use of the following lemmas:

Lemma 1.36

For $n \ge 2$,

$$\frac{1}{2\sqrt{n}} < \frac{B_n}{2^{2n}} < \frac{1}{\sqrt{2n}}.$$

Proof. Note that

$$\frac{B_n}{2^{2n}} = \frac{(2n)!}{(2^n n!)^2} = \frac{1\cdot 2\cdots 2n}{(2\cdot 4\cdots 2n)^2} = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}.$$

For the upper bound,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(2n)^2}\right) < 1,$$

as the product of elements in $(0,1)$. Note that

$$\begin{aligned}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(2n)^2}\right) &= \left(\frac{2^2-1}{2^2}\right)\left(\frac{4^2-1}{4^2}\right)\cdots\left(\frac{(2n)^2-1}{(2n)^2}\right)\\ &= \left(\frac{1\cdot 3}{2^2}\right)\left(\frac{3\cdot 5}{4^2}\right)\cdots\left(\frac{(2n-1)(2n+1)}{(2n)^2}\right)\\ &= (2n+1)\left(\frac{B_n}{2^{2n}}\right)^2.\end{aligned}$$

So

$$\frac{B_n}{2^{2n}} < \frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{2n}}.$$

For the lower bound,

$$\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\left(1-\frac{1}{(2n-1)^2}\right) < 1,$$

as before. Then

$$\begin{aligned}\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\left(1-\frac{1}{(2n-1)^2}\right) &= \left(\frac{3^2-1}{3^2}\right)\left(\frac{5^2-1}{5^2}\right)\cdots\left(\frac{(2n-1)^2-1}{(2n-1)^2}\right)\\ &= \left(\frac{2\cdot 4}{3^2}\right)\left(\frac{4\cdot 6}{5^2}\right)\cdots\left(\frac{(2n-2)(2n)}{(2n-1)^2}\right)\\ &= \frac{1}{2(2n)}\left(\left(\frac{B_n}{2^{2n}}\right)^{-1}\right)^2.\end{aligned}$$

So

$$\frac{B_n}{2^{2n}} > \frac{1}{\sqrt{4n}} = \frac{1}{2\sqrt{n}}. \qquad \blacksquare$$

Corollary 1.37
For $n \geq 1$,

$$\vartheta(n) < 2n \log 2.$$

Proof. We proceed by (an interesting form of) induction. This holds for $n = 1, 2$. We now show that if it holds for n, then it holds for $2n-1$ and $2n$.[1]

Consider the quantity

$$\frac{1}{2}B_n = \frac{1}{2}\binom{2n}{n} = \frac{(2n)!}{2n(n-1)!n!} = \frac{(2n-1)!}{(n-1)!n!} = \binom{2n-1}{n-1}.$$

We note that if $n < p < 2n$, then $p \mid \frac{1}{2}B_n$. Therefore,

$$\vartheta(2n-1) - \vartheta(n) = \prod_{n<p<2n} \log p \leq \log(B_n/2).$$

By induction hypothesis and Lemma 1.36,

$$\begin{aligned}\vartheta(2n-1) &< \log(B_n/2) + \vartheta(n)\\ &< \log(B_n/2) + 2n\log 2\\ &= \log(B_n) - \log 2 + 2n \log 2\\ &< 2n\log 2 - \frac{1}{2}\log 2n - \log 2 + 2n\log 2\\ &= 2n\log 2 - \frac{1}{2}(\log 2 + \log n) - \log 2 + 2n \log 2\\ &\leq 4n\log 2 - 2\log 2\\ &= 2(2n-1)\log 2.\end{aligned}$$

Since $2n$ is not prime for $n \geq 1$,

$$\vartheta(2n) = \vartheta(2n-1) < 2(2n-1)\log 2 < 2(2n)\log 2. \qquad \square$$

[1]Draw a tree of implications/try to prove that this induction scheme leads to a proof for all n.

Lemma 1.38
For $n \geq 8$,
$$\pi(n) \leq \frac{n}{2}.$$

Proof. For $n = 8, 9$, there are 4 primes: $2, 3, 5, 7$, which is less than or equal to half of the numbers up to n. For any $m > 4$, $\{2m, 2m+1\}$ contains at most 1 prime, $2m+1$. Therefore, it is impossible for more than half of the numbers in $\{1, \ldots, 2m\}$ or $\{1, \ldots, 2m+1\}$ to be prime. $\square$

Proof of Theorem 1.35. We are going improve our bounding technique for $\log(B_n)$ from the proof of Theorem 1.31. We have
$$\log(B_n) = \sum_{p \leq 2n} \sum_{k \geq 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) \log p.$$

We first eliminate terms in the inner sum where $k \geq 2$. If $p^2 > 2n$, then $\left\lfloor \frac{2n}{p^k} \right\rfloor = \left\lfloor \frac{n}{p^k} \right\rfloor = 0$, so there is no contribution. Therefore, we may split the sum into primes $\leq \sqrt{2n}$ and primes $\sqrt{2n} < p \leq 2n$. For the latter, we have the sum
$$\sum_{\sqrt{2n} < p \leq 2n} \left(\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \right) \log p.$$

Note that $\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor = 1$, if and only if $\frac{n}{p} \in [\frac{1}{2}, 1) + \mathbb{Z}$, if and only if
$$n \in [\frac{p}{2}, p) \cup [\frac{3p}{2}, 2p) \cup \cdots$$

Motivated by this, we split the sum into
$$\Sigma_1 = \sum_{\sqrt{2n} < p \leq \frac{2}{3}n} \left(\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \right) \log p,$$
$$\Sigma_2 = \sum_{\frac{2}{3}n < p \leq n} \left(\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \right) \log p,$$
$$\Sigma_3 = \sum_{n < p \leq 2n} \left(\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \right) \log p.$$

Since $\left(\left\lfloor \frac{2n}{p} \right\rfloor - 2\left\lfloor \frac{n}{p} \right\rfloor\right) \in \{0, 1\}$, we have the crude bound

$$\Sigma_1 \leq \vartheta\left(\frac{2n}{3}\right) - \vartheta(\sqrt{2n}) \leq \vartheta\left(\frac{2n}{3}\right) - \pi(\sqrt{2n})\log 2.$$

Since $\frac{2n}{3} < p \leq n \iff p \leq n < \frac{3}{2}p$, $\left\lfloor \frac{2n}{p} \right\rfloor - 2\left\lfloor \frac{n}{p} \right\rfloor = 0$ for all such p, so

$$\Sigma_2 = 0.$$

Finally, $n < p \leq 2n \iff \frac{p}{2} \leq n < p$, so $\left\lfloor \frac{2n}{p} \right\rfloor - 2\left\lfloor \frac{n}{p} \right\rfloor = 1$ for all such p. Therefore,

$$\Sigma_3 = \vartheta(2n) - \vartheta(n).$$

For the remaining terms, which we define as Σ_4,

$$\Sigma_4 = \sum_{p \leq \sqrt{2n}} \sum_{k \geq 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor\right) \log p,$$

we make use of the fact that once $k > \left\lfloor \frac{\log 2n}{\log p} \right\rfloor$, then $\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor = 0$. Then

$$\begin{aligned}
\Sigma_4 &\leq \sum_{p \leq \sqrt{2n}} \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \cdot \log p \\
&\leq \sum_{p \leq \sqrt{2n}} \frac{\log 2n}{\log p} \cdot \log p \\
&= \sum_{p \leq \sqrt{2n}} \log 2n \\
&= \pi(\sqrt{2n}) \log 2n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\log(B_n) &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 \\
&\leq \vartheta(2n) - \vartheta(n) + \vartheta\left(\frac{2n}{3}\right) - \pi(\sqrt{2n})(\log 2 - \log 2n).
\end{aligned}$$

We now show that this suffices to prove the claim for $n \geq 64$. [TO FINISH... some calculus] $\square$

2. Complex analysis preliminaries and more basic results

2.1. Infinite products

February 4, 2026

We say that $\prod_{n=1}^{\infty} a_n$, where $a_n \in \mathbb{C}$ **converges** if

$$\lim_{N\to\infty} \prod_{n=1}^{N} a_n$$

exists.

We say that $\prod_{n=1}^{\infty} a_n$ **converges absolutely** if for all bijections $\sigma\colon \mathbb{N} \to \mathbb{N}$, $\prod_{n=1}^{\infty} a_{\sigma(n)}$ converges.

Lemma 2.1

$\prod_{n=1}^{\infty} a_n$ converges absolutely $\iff$ $\sum_{n=1}^{\infty} |a_n - 1|$ converges.

Proof. ($\Longrightarrow$) It's easy to see that $\sum_n \log(a_n)$ also converges absolutely. [add proof?] Since $\log(a_n) \to 0$, $a_n \to 1$. Let $|a_n - 1|$ be sufficiently small so that

$$|\log(a_n)| \geq c|a_n - 1|.$$

Such a c exists by Taylor's approximation theorem. Therefore, comparing $\sum_n |\log(a_n)|$ and $\sum_n |a_n - 1|$, we have that the latter converges.

($\Longleftarrow$) We have that

$$\log(a_n) = \log(1 + (a_n - 1)) = (a_n - 1) - \frac{(a_n-1)^2}{2} + \frac{(a_n-1)^3}{3} + \cdots$$

We may assume that $|a_n - 1| < 1$ for all n by cutting off finitely many terms at the beginning. The above equation becomes

$$\log(a_n) = (a_n - 1) + O((a_n - 1)^2).$$

Therefore, $\sum_n |\log(a_n)|$ converges, so $\sum_n \log(a_n)$ converges absolutely. Let σ be any bijection of $\mathbb{N}$. Then $\sum_n \log(a_{\sigma(n)})$ converges. Exponentiating, we get that $\prod_n a_{\sigma(n)}$ converges, as desired. □

Definition 2.1

Let $U \subseteq \mathbb{C}$ be a *domain* (non-empty, open set), and $f_n, f\colon U \to \mathbb{C}$ be functions. We say $\sum_n f_n$ **converges normally** to f on $S \subseteq U$ if $\sum_n f_n \to f$ pointwise, and

$$\sum_{n=1}^{\infty} \sup\{|f_n(z)| : z \in S\} < \infty.$$

In particular, this condition implies that for all $z \in S$, $\sum_n f_n(z)$ converges absolutely, and that $\sum_{n=1}^{N} f_n$ converges uniformly to f on S.

Definition 2.2

Let $U \subseteq \mathbb{C}$ be a domain and $f_n, f\colon U \to \mathbb{C}$ be functions. We say $\sum_n f_n$ converges to f **locally normally** if for all $z \in U$, there is an open neighborhood $V_z \ni z$ such that $\sum_n f_n$ converges normally to f on V_z.

Equivalently, $\sum_n f_n$ converges locally normally to f if and only if for all $K \subseteq U$ compact, $\sum_n f_n$ converges normally to f on K.

Theorem 2.2 (Weierstrass)

If (f_n) is a sequence of holomorphic functions and $f = \sum_n f_n$ converges locally normally, then f is also holomorphic and

$$f' = \sum_n f_n',$$

with this convergence being locally normal. Moreover, if γ is a curve, then

$$\int_\gamma f(z)\,dz = \sum_n \int_\gamma f_n(z)\,dz.$$

Theorem 2.3 (Cauchy-Hadamard formula)

If $\sum_n c_n z^n$ has radius of convergence R, then

$$R^{-1} = \limsup_{n\to\infty} |c_n|^{1/n}.$$

Theorem 2.4 (Abel's convergence theorem)

If $\sum_n c_n z^n$ has radius of convergence R, then for all $r < R$, $\sum_n c_n z^n$ converges normally on $\overline{B}_r(0)$ and locally normally on $B_R(0)$.

Definition 2.3

If (f_n) is a sequence of holomorphic functions on $U \subseteq \mathbb{C}$, we say $\prod_n (1 - f_n(z))$ converges **locally normally** on U if $\sum_n f_n(z)$ converges locally normally.

When this condition is satisfied, then $\prod_n (1 - f_n(z))$ defines a holomorphic function $F(z)$ on U which is stable under rearrangement and satisfies

$$\log \prod_n (1 - f_n) = \sum_n \log(1 - f_n) \pmod{2\pi i \mathbb{Z}}$$

$$\frac{d}{dz} \prod_n (1 - f_n(z)) = \sum_{k=1}^{\infty} \prod_{n \neq k} (1 - f_n(z))(-f_k'(z)).$$

Note that these facts imply the log derivative is given by

$$\frac{F'(z)}{F(z)} = \sum_{k=1}^{\infty} \frac{-f_k'(z)}{1-f_k(z)} = \frac{d}{dz}\log(F(z)).$$

2.2. Abel summation

Theorem 2.5 (Abel)
Let $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be a sequence of real numbers such that $\lambda_n \to \infty$ and let (a_n) be a sequence of complex numbers. Let $A(x) = \sum_{\substack{n \\ \lambda_n \leq x}} a_n$ and let $\phi \colon \mathbb{R}_{\geq 0} \to \mathbb{C}$ be any function. Then

$$\sum_{n=1}^{k} a_n \phi(\lambda_n) = A(\lambda_k)\phi(\lambda_k) - \sum_{n=1}^{k-1} A(\lambda_n)\left(\phi(\lambda_{n+1}) - \phi(\lambda_n)\right).$$

Moreover, if ϕ has a continuous derivative on $(0, \infty)$, then

$$\sum_{\substack{n \\ \lambda_n \leq x}} a_n \phi(\lambda_n) = A(x)\phi(x) - \int_{\lambda_1}^{x} A(t)\phi'(t)\, dt.$$

Notice these expressions look a lot like integration by parts, hence why this technique is sometimes given the name *summation by parts*.

Corollary 2.6
If $A(x)\phi(x)$ in Theorem 2.5 tend to 0 as $x \to \infty$, then

$$\sum_{n=1}^{\infty} a_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} A(t)\phi'(t)\, dt.$$

2.3. Applying this to $\zeta(s)$

Theorem 2.7
The log derivative of the Riemann zeta function satisfies

$$-\frac{\zeta'(s)}{\zeta(s)} = s \cdot \int_{1}^{\infty} \frac{\psi(x)}{x^{s+1}}\, dx.$$

Proof. Recall that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Set $\lambda_n \coloneqq n$, $a_n \coloneqq \Lambda(n)$, and $\phi(x) \coloneqq x^{-s}$ in Theorem 2.5. In the notation of

Theorem 2.5, we have that $A(x) = \psi(x)$. Note that $\psi(x) \cdot x^{-s} \to 0$ as $x \to \infty$ for $\mathrm{Re}(s) > 1$. Indeed,

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \overset{(1.30,1.31)}{\leq} 4\log 2$$

(another way to see this is that $\psi(x) \leq \pi(x)\log x < x\log x$). Then apply Corollary 2.6. ☐

2.4. Mellin transforms

February 6, 2026

Definition 2.4

Let $f\colon \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous funtion. For $s \in \mathbb{C}$, the integral (if it converges)

$$\int_0^\infty f(t)t^s \,\frac{dt}{t}$$

is called the **Mellin transform** of f at s.

Remark 2.8. If we consider $\mathbb{R}_{>0}$ as a semigroup under multiplication, then $\frac{dt}{t}$ is an invariant measure on this semigroup. This is analogous to dt being an invariant measure for $\mathbb{R}$ under addition.

For a fixed $s \in \mathbb{C}$, the function $t \mapsto t^s = e^{s\log t}$ is a *quasi-character*, i.e., a homomorphism from $\mathbb{R}_{>0}$ to $\mathbb{C}^\times$. Therefore, one considers the Mellin transform as analogous to the Fourier transform but for the semigroup $(\mathbb{R}_{>0}, \times)$.

Example 2.1 –

$$-\frac{1}{s}\cdot\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx = \int_0^\infty \psi(x)x^{-s}\,\frac{dx}{x}.$$

In other words, $-\frac{1}{s}\cdot\frac{\zeta'(s)}{\zeta(s)}$ is the Mellin transform of ψ at $-s$.

Theorem 2.9

$$\liminf_{x\to\infty}\frac{\pi(x)}{x/\log x} \leq 1 \leq \limsup_{x\to\infty}\frac{\pi(x)}{x/\log x}.$$

Hence, if a limit exists, then it equals 1 (this is the prime number theorem).

Proof of Theorem 2.9. By Lemma 1.30, it suffices to show

$$\liminf_{x\to\infty}\frac{\psi(x)}{x} \leq 1 \leq \limsup_{x\to\infty}\frac{\psi(x)}{x}.$$

Set
$$\ell := \liminf_{x\to\infty} \frac{\psi(x)}{x}, \qquad L := \limsup_{x\to\infty} \frac{\psi(x)}{x}.$$

Let $f(s) := -\frac{\zeta'(s)}{\zeta(s)}$ be the log derivative of ζ and let
$$\ell' := \liminf_{s\to 1}(s-1)f(s), \qquad L' := \limsup_{s\to 1}(s-1)f(s).$$

To prove the result, we show that $\ell \le \ell'$ and $L' \le L$, then show that $\ell' = L' = 1$. We first show $L' \le L$. For any $B > L$, we have $\psi(x)/x < B$ for all sufficiently large $x \ge x_0(B) > 1$, where $x_0(B)$ is some constant. For $s > 1$,

$$\begin{aligned}
f(s) &= s \cdot \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx \\
&< s \cdot \int_1^{x_0(B)} \frac{\psi(x)}{x^{s+1}}\, dx + s \cdot \int_{x_0(B)}^\infty \frac{\psi(x)}{x^{s}}\, dx \\
&< s \cdot \int_1^{x_0(B)} \frac{\psi(x)}{x^{2}}\, dx + s \cdot \int_{x_0(B)}^\infty \frac{B}{x^{s}}\, dx \\
&= s \cdot \int_1^{x_0(B)} \frac{\psi(x)}{x^{2}}\, dx + \frac{Bs}{s-1}.
\end{aligned}$$

Noting that the integral is a constant only depending on B, call it K, we have
$$f(s)(s-1) < s(s-1)K + Bs.$$

Taking $s \to 1^+$ we get $L' \le B$. Since B was arbitrary, $L' \le L$.

We now show $\ell \le \ell'$. Similarly, let $b < \ell$, so $\psi(x)/x > b$ for all sufficiently large $x \ge x_0(b) > 1$. Then

$$\begin{aligned}
f(s) &= s \cdot \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx \\
&= s \cdot \int_1^{x_0(b)} \frac{\psi(x)}{x^{s+1}}\, dx + s \cdot \int_{x_0(b)}^\infty \frac{\psi(x)}{x^{s+1}}\, dx \\
&\ge s \cdot \int_{x_0(b)}^\infty \frac{\psi(x)}{x^{s}}\, dx \\
&= \frac{s}{1-s} \cdot b \cdot x_0(b)^{-s+1}.
\end{aligned}$$

Rearranging and taking $s \to 1^+$, we get $\ell' \ge b$. Since b was arbitrary, $\ell \le \ell'$.

We now show that $\ell' = L' = 1$. Consider the limit
$$\lim_{s\to 1^+} -(s-1)^2\zeta'(s).$$

For $s > 1$ and $x \geq e$, $x^{-s}\log x$ is decreasing, so

$$\zeta'(s) = \sum_{n=1}^{\infty} \frac{\log n}{n^s} = \int_1^\infty \frac{\log x}{x^s}\,dx + O(1).$$

We perform a change of variables with $x^{s-1} = e^y$, so $y = (s-1)\log x$, $dy = (s-1)\frac{dx}{x}$, yielding

$$\int_1^\infty \frac{\log x}{x^s}\,dx = \int_0^\infty \frac{1}{(s-1)^2}\cdot\frac{y}{e^y}\,dy = \frac{1}{(s-1)^2}\cdot\int_0^\infty ye^{-y}\,dy = \frac{1}{(s-1)^2}.$$

Therefore, the limit described above is

$$\lim_{s\to 1^+} -(s-1)^2\cdot\left(\frac{1}{(s-1)^2} + O(1)\right) = 1.$$

We also consider

$$\lim_{s\to 1^+}(s-1)\zeta(s).$$

For $s > 1$, x^{-s} is decreasing so

$$\int_1^\infty \frac{1}{x^s}\,dx < \sum_{n=1}^{\infty}\frac{1}{n^s} < 1 + \int_1^\infty \frac{1}{x^s}\,dx.$$

Thus,

$$\frac{1}{s_1} < \zeta(s) < \frac{s}{s-1},$$

which implies

$$\lim_{s\to 1^+}(s-1)\zeta(s) = 1.$$

Combining these facts, we have that

$$\lim_{s\to 1^+}(s-1)f(s) = \lim_{s\to 1^+}\frac{-\zeta'(s)(s-1)^2}{\zeta(s)(s-1)} = 1. \qquad \square$$

2.5. Some formulae of Mertens

Theorem 2.10 (Mertens)

As $x \to \infty$,

1. $\sum_{n\leq x}\frac{\Lambda(n)}{n} = \log x + O(1)$.
2. $\sum_{p\leq x}\frac{\log p}{p} = \log x + O(1)$.
3. $\int_1^\infty \frac{\psi(t)}{t^2}\,dt = \log x + O(1)$
4. $\sum_{p\leq x}\frac{1}{p} = \log\log x + C + O\left(\frac{1}{\log x}\right)$.

Proof. (1) Chebyshev's inequality (1.31)

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \le 4\log 2$$

implies $\psi(x) = O(x)$. Recall that

$$\log m! = \sum_{p^r \le m} \left\lfloor \frac{m}{p^r} \right\rfloor \log p = \sum_{n\le m} \left\lfloor \frac{m}{n} \right\rfloor \Lambda(n).$$

Let $\frac{m}{n} = \left\lfloor \frac{m}{n} \right\rfloor + \varepsilon_n$. Then

$$\begin{aligned} \log m! &= \sum_{n\le m} \frac{m}{n}\Lambda(n) + \sum_{n\le m} \varepsilon_n \Lambda(n) \\ &\le \sum_{n\le m} \frac{m}{n}\Lambda(n) + \psi(m) \\ &= m \sum_{n\le m} \frac{\Lambda(n)}{n} + O(m). \end{aligned}$$

Dividing by m we get[1]

$$\sum_{n\le m} \frac{\Lambda(n)}{n} = \frac{\log m!}{m} + O(1) = \log m + O(1).$$

If x is an arbitrary positive real number, then

$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \log\lfloor x\rfloor + O(1) = \log x + O(1),$$

since, for example, $\log\lfloor x\rfloor \le \log 2x = \log x + O(1)$.

(2)

$$\begin{aligned} \left| \sum_{n\le x} \frac{\Lambda(n)}{n} - \sum_{p\le x} \frac{\log p}{p} \right| &\le \sum_{p\le x} \log p \left(\frac{1}{p^2} + \frac{1}{p^3} + \cdots \right) \\ &= \sum_{p\le x} \frac{\log p}{p(p-1)} \\ &< \sum_{n\ge 1} \frac{\log n}{n(n-1)} < \infty. \end{aligned}$$

Combining this with (1) gives the result.

(3)

$$\begin{aligned}\int_1^x \frac{\psi(t)}{t^2}\,dt &= \int_1^x \sum_{n\le t} \frac{\Lambda(n)}{t^2}\,dt\\ &= \sum_{n\le x} \Lambda(n) \int_n^x \frac{1}{t^2}\,dt\\ &= \sum_{n\le x} \Lambda(n) \left[-\frac{1}{t}\right]_n^x\\ &= \sum_{n\le x} \Lambda(n) \left(\frac{1}{n}-\frac{1}{x}\right)\\ &= \sum_{n\le x} \frac{\Lambda(n)}{n} - \frac{\sum_{n\le x}\Lambda(n)}{x}\\ &= \log x + O(1) - \frac{\psi(x)}{x}\\ &= \log x + O(1).\end{aligned}$$

(4) Consider Abel summation with $\lambda_n = p_n$, $a_n = \frac{\log p_n}{p_n}$, and $\phi(x) = \frac{1}{\log x}$. We then have (using (2))

$$\begin{aligned}\sum_{p\le x}\frac{1}{p} &= \left(\sum_{p\le x}\frac{\log p}{p}\right)\frac{1}{\log x} + \int_2^x \left(\sum_{p\le t}\frac{\log p}{p}\right)\frac{1}{t\log^2 t}\,dt\\ &= 1 + O\left(\frac{1}{\log x}\right) + \int_2^x (\log t + O(1))\frac{1}{t\log^2 t}\,dt\\ &= 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{1}{t\log t}\,dt + O(1)\cdot\int_2^x \frac{1}{t\log^2 t}\,dt\\ &= 1 + O\left(\frac{1}{\log x}\right) + \log\log x - \log\log 2 + O(1)\cdot\left(\frac{1}{\log x} - \frac{1}{\log 2}\right)\\ &= \log\log x + 1 - \log\log 2 + O\left(\frac{1}{\log x}\right).\end{aligned}$$

□

[1] The last equality uses the following fact that we proved before:

$$\log m! \le m\log m - m + 1 + \frac{1}{2}\log m = m\log m + O(m).$$

3. (non-Elementary) Analytic Number Theory

3.1. Extending ζ to $\sigma > 0$

February 20, 2026 Recall that we showed that $\zeta(s) = \zeta(\sigma + it)$, converges when $\sigma > 1$. We will now make it analytic on $\sigma > 0$, except for a simple pole at $s = 1$ with residue 1.

By Abel summation (2.5) with $\lambda_n = n$, $\phi(x) = x^{-s}$, $a_n = 1$, we have $A(x) = \lfloor x \rfloor$, so

$$\sum_{n \le x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor u \rfloor}{u^{s+1}} \, du.$$

Since $\lfloor x \rfloor / x^s \to 0$ as $x \to \infty$ and $\sum_n \frac{1}{n^{-s}}$ converges when $\sigma > 1$, letting $u = \lfloor u \rfloor + \{u\}$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = s \int_1^{\infty} \frac{1}{u^s} \, du - s \int_1^{\infty} \frac{\{u\}}{u^{s+1}} \, du = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{u\}}{u^{s+1}} \, du.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{s-1} - s \int_1^{\infty} \frac{\{u\}}{u^{s+1}} \, du$$

on $\sigma > 1$. But since $0 \le \{u\} < 1$, the integral absolutely and uniformly converges on every half-plane $\sigma \ge \delta > 0$ for fixed δ. We prove that the integral is analytic using Morera's theorem. Let γ be a triangle in $\sigma \ge \delta > 0$. On the triangle, we have that

$$\left| \frac{\{u\}}{u^{s+1}} \right| \le \frac{1}{|u|^{\delta+1}}.$$

Therefore

$$\oint_{\gamma} \int_1^{\infty} \frac{\{u\}}{u^{s+1}} \, du \, ds$$

absolutely converges, so we may interchange integrals to get

$$\int_1^{\infty} \oint_{\gamma} \frac{\{u\}}{u^{s+1}} \, du \, ds.$$

Since for fixed u, $\frac{\{u\}}{u^{s+1}}$ is holomorphic on $\sigma > 0$, the inner integral is zero, so the entire integral is zero.

3.2. Dirichlet series

March 6, 2026

Definition 3.1

A **Dirichlet series** is a series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ where $a_n \in \mathbb{C}$.

Theorem 3.1
If $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $s = s_0$, then it converges uniformly in the region

$$\left\{ s : |\arg(s - s_0)| \leq \frac{\pi}{2} - \theta < \frac{\pi}{2} \right\}$$

for a fixed $\theta \in \left(0, \frac{\pi}{2}\right)$.

Proof. By replacing the series by $\sum_{n=1}^{\infty} a_n n^{-s_0}/n^s$, we may assume that $s_0 = 0$. That is, $\sum_n a_n$ converges.

Fix $M < N$. By Abel's theorem, taking $\phi(t) = t^{-s}$, we have

$$\begin{aligned}\sum_{n=M}^{N} a_n n^{-s} &= A(N)\phi(N) - \int_M^N A(t)\phi'(t)\,dt \\ &= \left(\sum_{n=M}^{N} a_n\right) N^{-s} - s\int_M^N \frac{A(t)}{t^{s+1}}\,dt.\end{aligned}$$

For $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that for all $N > M \geq n_0$,

$$\left|\sum_{n=M}^{N} a_n\right| < \varepsilon.$$

Then

$$\begin{aligned}\left|\sum_{n=M}^{N} a_n n^{-s}\right| &\leq \frac{\varepsilon}{N^\sigma} + |s|\int_M^N \frac{\varepsilon}{t^\sigma}\,dt \\ &= \frac{\varepsilon}{N^\sigma} + |s| \cdot \varepsilon \cdot \frac{1}{\sigma} \cdot \left(\frac{1}{N^\sigma} - \frac{1}{M^\sigma}\right) \\ &\leq \frac{\varepsilon}{N^\sigma} + \frac{|s|}{\sigma} \cdot \varepsilon \cdot \frac{1}{M^\sigma} \\ &= \frac{\varepsilon}{N^\sigma} + \frac{1}{\cos(\arg s)} \cdot \varepsilon \cdot \frac{1}{M^\sigma} \\ &= \frac{\varepsilon}{N^\sigma} + \frac{1}{\sin\left(\frac{\pi}{2} - |\arg s|\right)} \cdot \varepsilon \cdot \frac{1}{M^\sigma} \\ &\leq \frac{\varepsilon}{N^\sigma} + \frac{1}{\sin\theta} \cdot \varepsilon \cdot \frac{1}{M^\sigma} \\ &\leq \frac{2}{M^\sigma} \cdot \frac{1}{\sin\theta} \cdot \varepsilon \xrightarrow{M \to \infty} 0.\end{aligned}$$

□

To see what this region looks like, consider Figure 2. Note that as we take $\theta \to \frac{\pi}{2}$ we get convergence in a half plane. We define

$$\sigma_0 := \inf\left\{ \operatorname{Re} s : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges} \right\},$$

where we take the convention $\inf \varnothing = \infty$. Note that:

- If $\operatorname{Re} s < \sigma_0$, then $\sum_{n=1}^{\infty} a_n n^{-s}$ diverges.
- If $\sigma_0 = -\infty$, then $\sum_{n=1}^{\infty} a_n n^{-s}$ converges everywhere.

We call σ_0 the **abscissa of convergence**.

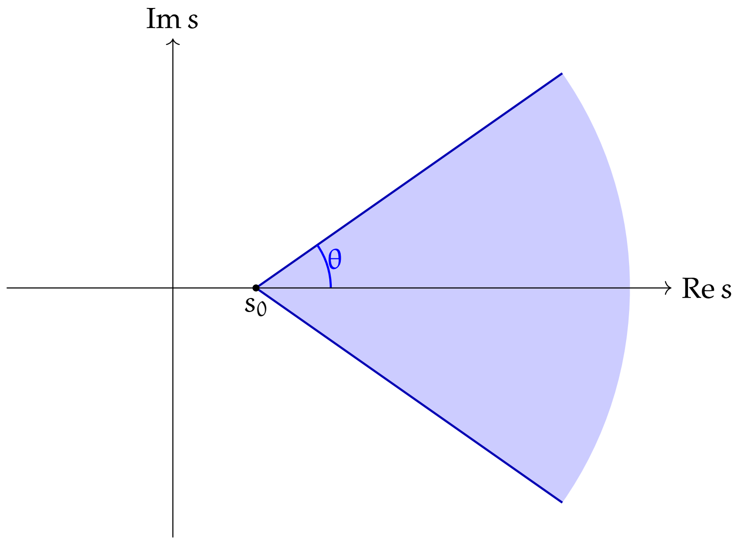


Figure 2: Region in Theorem 3.1. Made with help from GitHub Copilot.

Theorem 3.2

Let σ_0 be the abscissa of convergence for the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$. Then

1. $\sum_{n=1}^{\infty} a_n n^{-s}$ is a holomorphic function on $\sigma > \sigma_0$.
2. The derivatives can be computed termwise in the region $\sigma > \sigma_0$. For example, $\frac{d}{ds}\sum_{n=1}^{\infty} a_n n^{-s} = -\sum_{n=1}^{\infty} a_n (\log n) n^{-s}$.

Proof. (1) If γ is a closed loop in $\{s : \operatorname{Re} s > \sigma_0\}$, there exists a θ such that $\gamma \subseteq \left\{s : |\arg(s - \sigma_0) \leq \frac{\pi}{2} - \theta\right\}$ by compactness. Because we have uniform convergence in this region, we apply Fubini:

$$\int_\gamma \sum_{n=1}^{\infty} a_n n^{-s}\, ds = \sum_{n=1}^{\infty} \int_\gamma a_n n^{-s}\, ds = 0.$$

By Morera's theorem, the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is holomorphic.

(2) For any holomorphic function f, Cauchy's differentiation formula gives

$$f'(s) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - s)^2}\, d\xi,$$

where γ is a loop around s. In particular, if we take $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, we have, for γ a sufficiently small circle around s with $\operatorname{Re} s = \sigma > \sigma_0$,

$$\begin{aligned} f'(s) &= \frac{1}{2\pi i} \int_\gamma \frac{\sum_{n=1}^\infty a_n n^{-\xi}}{(\xi - s)^2} \, d\xi \\ &= \sum_{n=1}^\infty \frac{1}{2\pi i} \int_\gamma \frac{a_n n^{-\xi}}{(\xi - s)^2} \, d\xi \\ &= \sum_{n=1}^\infty a_n \frac{d}{ds} n^{-s} \\ &= -\sum_{n=1}^\infty a_n (\log n) n^{-s}, \end{aligned}$$

where swapping the sum and the integral is justified by the fact that $\frac{1}{(\xi - s)^2}$ is bounded away from zero on γ by compactness, and therefore the series $\sum_{n=1}^\infty \frac{a_n n^{-\xi}}{(\xi - s)^2}$ converges uniformly on γ. □

Definition 3.2
The **abscissa of absolute convergence** for $\sum_{n=1}^\infty a_n n^{-s}$, denoted $\tilde{\sigma}_0$, is the abscissa of convergence for $\sum_{n=1}^\infty |a_n| n^{-s}$.

Obviously, $\tilde{\sigma}_0 \geq \sigma_0$. Let the **strip of conditional convergence** be the region $\{s : \sigma_0 < \sigma < \tilde{\sigma}_0\}$.

Theorem 3.3
For any Dirichlet series, $\tilde{\sigma}_0 - \sigma_0 \leq 1$.

Proof. If $\sum_{n=1}^\infty a_n n^{-s}$ converges, then $|a_n n^{-s}| \xrightarrow{n \to \infty} 0$, so for $\varepsilon > 0$,

$$\sum_{n=1}^\infty \frac{|a_n|}{n^{1+s+\varepsilon}} = \sum_{n=1}^\infty \frac{|a_n|}{n^s} \cdot \frac{1}{n^{1+\varepsilon}},$$

which converges because $|a_n n^{-s}|$ is bounded and $1 + \varepsilon > 1$. □

Theorem 3.4
If $\sum_{n=1}^\infty a_n n^{-s}$ and $\sum_{n=1}^\infty b_n n^{-s}$ converge on $\sigma > \sigma_0$, and they are equal on the same open subset of this half-plane, then $a_n = b_n$ for all n.

Remark 3.5. For example, if $\sum_n |a_n| < \infty$, then for any bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$ the Dirichlet series $\sum_n a_{\sigma(n)} n^{-s}$ agrees with $\sum_n a_n n^{-s}$ at $s = 0$, but the coefficients may differ. We therefore see that the assumption of equality on an open subset cannot be removed.

Proof of Theorem 3.4. Let $c_n = a_n - b_n$, and consider $\sum_{n \geq 1} c_n n^{-s}$. This Dirichlet series is a 0 on $\sigma > \sigma_0$, since it is zero on an open subset and both Dirichlet series are holomorphic on $\sigma > \sigma_0$. Suppose $M > 0$ is the minimal such index so that $c_M \neq 0$. Then

$$\sum_{n=1}^{\infty} c_n n^{-s} = \sum_{n=M}^{\infty} c_n n^{-s} = 0,$$

so

$$c_M M^{-s} = -\sum_{n=M+1}^{\infty} c_n n^{-s}, \qquad \text{when } \sigma > \sigma_0.$$

If $\sigma > \sigma_0 + 1$, then

$$\left|c_M M^{-s}\right| \leq \sum_{n=M+1}^{\infty} |c_n n^{-s}| \implies |c_M| \leq \sum_{n=M+1}^{\infty} |c_n| \cdot \left(\frac{M}{n}\right)^{\sigma}.$$

Let $\varepsilon > 0$. Then for $\sigma \geq \sigma_0 + 1 + \varepsilon$ we have uniform convergence, so

$$\lim_{\sigma \to \infty} \sum_{n=M+1}^{\infty} |c_n| \cdot \left(\frac{M}{n}\right)^{\sigma} = \sum_{n=M+1}^{\infty} \lim_{\sigma \to \infty} |c_n| \cdot \left(\frac{M}{n}\right)^{\sigma} = 0.$$

This implies that $|c_M| \leq 0$, contradiction. $\square$

Theorem 3.6 (Landau)
Let $f(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series such that $a_n \geq 0$ for all n and σ_0 is finite, then $s = \sigma_0$ is a singularity for the Dirichlet series.

Proof. Since $a_n \geq 0$ for all n, the abscissa of convergence and absolute convergence agree. We may also assume that $\sigma_0 = 0$ by replacing a_n with $a_n n^{-\sigma_0}$. Suppose that $s = \sigma_0$ is not a singularity. Then there exists an open ball around σ_0 so that $\sum_n a_n n^{-s}$ admits a holomorphic extension to the union of $\{s : \sigma > 0\}$ and the ball. So the radius of convergence of

$$f(s) = \sum_{\nu=0}^{\infty} \frac{(s-1)^{\nu}}{\nu!} f^{(\nu)}(1)$$

is $\rho > 1$ (see Figure 3).

For $s > 0$, $f(s) = \sum_{n=1}^{\infty} a_n e^{-(\log n) \cdot s}$, so

$$f^{(\nu)}(s) = \sum_{n=1}^{\infty} a_n (-\log n)^{\nu} e^{-(\log n) \cdot s},$$

which implies at $s = 1$,

$$f^{(\nu)}(1) = \sum_{n=1}^{\infty} \frac{a_n(-\log n)^\nu}{n},$$

so

$$f(s) = \sum_{\nu=0}^{\infty} \frac{(s-1)^\nu}{\nu!} \sum_{n=1}^{\infty} \frac{a_n(-\log n)^\nu}{n} = \sum_{\nu=0}^{\infty} \frac{(1-s)^\nu}{\nu!} \sum_{n=1}^{\infty} \frac{a_n(\log n)^\nu}{n}.$$

If $s < 0$, the above sum converges and (almost) all terms are ≥ 0, so it actually converges absolutely. Reordering,

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{\nu=0}^{\infty} \frac{(1-s)^\nu(\log n)^\nu}{\nu!} = \sum_{n=1}^{\infty} \frac{a_n}{n} e^{(1-s)\log n} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

must converge for some $s < 0$, which contradicts $\sigma_0 = 0$. □

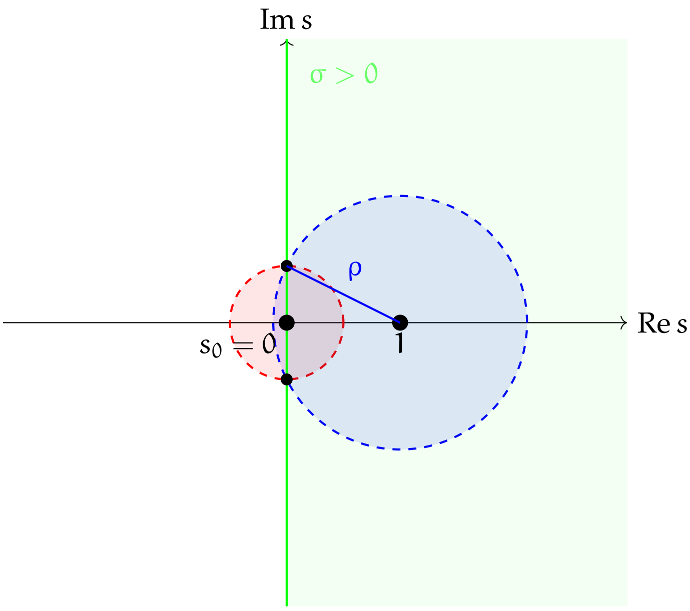


Figure 3: Visualization for the proof of Theorem 3.6 showing why a $\rho > 1$ exists if f is holomorphic around $s = 0$. Made with help from GitHub Copilot.

Definition 3.3

March 11, 2026

If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ are Dirichlet series, then define their **formal product** as the sum

$$h(s) = \sum_{n=1}^{\infty} \left(\sum_{k\ell=n} a_k b_\ell \right) n^{-s}.$$

If f, g both converge on $\sigma > \sigma_0$, then $h(s) = f(s) \cdot g(s)$ on $\sigma > \sigma_0 + 1$.

Example 3.1 – Consider the Dirichlet series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$. We claim it equals $\left(1-2^{1-s}\right)\zeta(s)$ when $\sigma > 0 = \sigma_0$. If $\sigma > 1$, then we have absolute convergence so we may rearrange the sum to get

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} &= \frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\cdots \\ &= \left(\frac{1}{1^s}+\frac{1}{3^s}+\cdots\right)-\left(\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\cdots\right) \\ &= \left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)-2\left(\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\cdots\right) \\ &= \zeta(s)-2\cdot 2^{-s}\cdot\zeta(s) \\ &= \zeta(s)(1-2^{1-s}).\end{aligned}$$

By the uniqueness of analytic continuation, equality holds on $\sigma > 0$.

3.3. Dirichlet's theorem

For this section, I assume basic knowledge of character theory, in particular the orthogonality relations.

Let χ be a Dirichlet character modulo m. That is, $\chi\colon (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ is a homomorphism, and we extend χ to a function on $\mathbb{Z}$ by letting $\chi(k) = \chi(k \mod m)$ if k is a unit modulo m, and 0 otherwise. The associated **Dirichlet series** to χ is the function

$$L(s,\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

Since $\left|\frac{\chi(n)}{n^s}\right| \le \frac{1}{n^\sigma}$, so $L(s,\chi)$ converges locally normally on $\sigma > 1$.

For a fixed m, we consider $L(s,\chi)$ for $\chi = \chi_{\text{triv}}$ (which is 1 if $(k,m) = 1$ and 0 otherwise) and $\chi \neq \chi_{\text{triv}}$.

- **(Case 1:** $\chi \neq \chi_{\text{triv}}$**)** Consider $A(x) := \sum_{n\le x}\chi(n)$. We have that $\chi(1)+\cdots+\chi(m) = 0$, so $|A(x)| \le m$. Applying Abel summation to $\lambda_n = n$, $a_n = \chi(n)$ and $\phi(t) = \frac{1}{t^s}$, we get

$$\sum_{n\le x}\frac{\chi(n)}{n^s} = \underbrace{A(x)\phi(x)}_{\substack{\to 0 \\ \text{as } x\to\infty \text{ when } \sigma>0}} - \int_1^x A(t)\phi'(t)\,dt.$$

So

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} = s\int_1^{\infty}\frac{A(t)}{t^{s+1}}\,dt.$$

The integral converges on $\sigma > 0$ because $\left|\frac{A(t)}{t^{s+1}}\right| \le \frac{m}{t^{\sigma+1}}$. Therefore, $\sigma_0 \le 0$ for $L(s,\chi)$. On the other hand, it is clear that $L(s,\chi)$ diverges for $\sigma < 0$, so $\sigma_0 = 0$.

- **(Case 2:** $\chi = \chi_{\text{triv}}$) Recall that $\chi(n)$ is 1 if $(n, m) = 1$ and 0 otherwise. Then

$$\begin{aligned} L(s,\chi) &= \sum_{\substack{n \\ (n,m)=1}} \frac{1}{n^s} \\ &= \prod_{\substack{p \text{ prime} \\ p\nmid m}} \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \prod_{p} \left(1 - \frac{1}{p^s}\right) - 1 \cdot \prod_{p|m} \left(1 - \frac{1}{p^s}\right) \\ &= \zeta(s) \cdot \prod_{p|m} \left(1 - \frac{1}{p^s}\right). \end{aligned}$$

  Note that $\prod_{p|m} \left(1 - \frac{1}{p^s}\right)$ is holomorphic around $s = 1$, whereas $\zeta(s)$ has a pole at $s = 1$. Therefore, $L(s, \chi)$ has a pole at $s = 1$, meaning $\sigma_0 \geq 1$. On the other hand, $L(s, \chi)$ converges on $\sigma > 1$ because the abscissa of convergence of $\zeta(s)$ is 1, so $\sigma_0 = 1$.

Note that the existence of an analytic continuation of $\zeta(s)$ to $\sigma > 0$ (except a pole at $s = 1$) implies that $L(s, \chi_{\text{triv}})$ also has such an extension. Therefore, we may compute

$$\operatorname{Res}_{s=1} L(s, \chi_{\text{triv}}) = 1 \cdot \prod_{p|m} \left(1 - \frac{1}{p^s}\right) = \frac{\phi(m)}{m}.$$

Lemma 3.7

For all $\chi \neq \chi_{\text{triv}}$, $L(1, \chi) \neq 0$.

The idea is that $\prod_\chi L(s, \chi)$ is holomorphic near $s = 1$ if any of the Dirichlet series of the nontrivial characters are zero at 1.

Proof of Lemma 3.7. Define

$$\log L(s,\chi) := \sum_p -\log(1-p^{-s}) = \sum_{p,k} \frac{\chi(p^k)}{kp^{ks}}.$$

With the appropriate log branch for the second sum, $e^{\log L(s,\chi)} = L(s,\chi)$.

Let $Q(s) = \sum_\chi \log L(s, \chi)$. Our first goal is to show that $Q(s)$ is a Dirichlet series.

Note that

$$\begin{aligned} Q(s) &= \sum_{k,p} \sum_{\chi} \frac{\chi(p^k)}{kp^{ks}} \\ &= \sum_{k,p} \frac{1}{kp^{ks}} \sum_{\chi} \chi(p^k) \\ &= \sum_{\substack{k,p \\ p^k \equiv 1(m)}} \frac{1}{kp^{ks}} \cdot \phi(m), \end{aligned}$$

where the last line is by the orthogonality relations. Set

$$a_n := \begin{cases} \phi(m)/k & \text{if } n = p^k \equiv 1 \pmod{m}, \\ 0 & \text{otherwise,} \end{cases}$$

so $Q(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a Dirichlet series and converges for $\sigma > 1$. Let $h = \phi(m)$.

If $p \nmid m$, then $p^h \equiv 1 \pmod{m}$. Therefore,

$$Q(s) > \sum_{p \nmid m} \frac{1}{p^{hs}} = \sum_{p} \frac{1}{p^{hs}} - \sum_{p \mid m} \frac{1}{p^{hs}}.$$

If $s = \frac{1}{h}$, then the RHS diverges to ∞, so $\sigma_0 \in \left[\frac{1}{h}, 1\right]$. Now consider the following function defined on $\sigma > 1$:

$$P(s) = \exp(Q(s)) = 1 + Q(s) + \frac{Q(s)^2}{2!} + \cdots.$$

Note that $P(s)$ converges at least on the half-plane that $Q(s)$ converges on. By gathering terms, we know that $P(s)$ is also a Dirichlet series. Therefore, the abscissa of convergence for $P(s)$ (σ_0^P) is less than or equal to the abscissa of convergence for $Q(s)$ (σ_0^Q). For $s > \sigma_0^P$, we note that the Dirichlet coefficients of $P(s)$ are greater than or equal to the corresponding Dirichlet coefficients of $Q(s)$ for each n. Therefore, by comparison, $Q(s)$ must converge. This yields $\sigma_0^P = \sigma_0^Q$.

We have the equality

$$P(s) = \prod_{\chi} L(s, \chi).$$

If there exists a character χ such that $L(1, \chi) = 0$, then P is holomorphic on $\sigma > 0$, which contradicts Landau's theorem (3.6). □

Theorem 3.8 (Dirichlet's theorem)

If $m, a \in \mathbb{Z}_{\geq 0}$ are such that $(a, m) \neq 1$, then there are infinitely many primes p such that $p \equiv a \pmod{m}$.

Proof. We have that

$$\log L(s,\chi) = \sum_{k\geq 1}\sum_p \frac{\chi(p^k)}{kp^{ks}} = \sum_p \frac{\chi(p)}{p^s} + \underbrace{\sum_{k=2}^{\infty}\sum_p \frac{\chi(p^k)}{kp^{ks}}}_{(1)}.$$

We claim that (1) converges when $\sigma = \operatorname{Re} s > \frac{1}{2}$. Since $|\chi(p^k)| = 1$, it suffices to show that

$$\sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{kn^{k\sigma}} < \infty.$$

Since all terms are positive, it is equivalent to show

$$\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{kn^{k\sigma}} < \infty.$$

We have that

$$\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{kn^{k\sigma}} = \sum_{n=2}^{\infty} -\log(1-n^{-\sigma}) - n^{-\sigma}.$$

If $u \in \left(0, \frac{1}{2}\right)$,

$$\begin{aligned} -\log(1-u) - u &\leq \frac{u^2}{2} + \frac{u^3}{3} + \cdots \\ &\leq \frac{u^2}{2} + \frac{u^3}{2} + \cdots \\ &= \frac{u^2}{2(1-u)} \\ &\leq u^2. \end{aligned}$$

For n sufficiently large, $n^{-\sigma} \in \left(0, \frac{1}{2}\right)$, so the sum $\sum_{n=2}^{\infty} -\log(1-n^{-\sigma}) - n^{-\sigma}$ converges if $\sum_{n=1}^{\infty} n^{-2\sigma}$ converges. But the latter is $\zeta(2\sigma)$, and $2\sigma > 1$, so our original sum converges.

Because of this, define (1) for fixed χ as $R(s,\chi) < \infty$, and define $R^*(s) \coloneqq \sum_\chi R(s,\chi)\chi(h)$. Let $b \in \mathbb{Z}_{\geq 0}$ such that $ab \equiv 1 \pmod m$. Then

$$\begin{aligned} \sum_\chi \chi(b)\log L(s,\chi) &= \sum_\chi \sum_p \frac{\chi(pb)}{p^s} + R^*(s) \\ &= \sum_p \sum_\chi \frac{\chi(pb)}{p^s} + R^*(s) \\ &= \sum_{pb\equiv 1(m)} \frac{h}{p^s} + R^*(s) \\ &= \sum_{p\equiv a(m)} \frac{h}{p^s} + R^*(s). \end{aligned}$$

We know that as $s \to 1^+$, $L(s, \chi_{\text{triv}}) \to \infty$. For $\chi \neq \chi_{\text{triv}}$, $L(s,\chi) \not\to 0$ as $s \to 1^+$ by Lemma 3.7. To establish that $\log L(s,\chi)$ converges as $s \to 1^+$, we have the formula

$$\log L(s,\chi) = -\int_s^c \frac{L'(u,\chi)}{L(u,\chi)}\, du + \log L(c,\chi)$$

for real $s, c \in B_\delta(1)$ and $c > s$ for a sufficiently small δ so that $L(u,\chi) \neq 0$ on all $u \in B_\delta(1)$. This proves the fact because $\frac{L'(u,\chi)}{L(u,\chi)}$ is bounded on a sufficiently small neighborhood around 1 because $L'(s,\chi)$ is analytic on $\sigma > 0$ and $L(s,\chi)$ is nonzero around $s = 1$.

Since $R^*(s)$ is analytic for $\sigma > \frac{1}{2}$, the sum $\sum_\chi \chi(b) \log L(s,\chi)$ diverges as $s \to 1^+$, so $\sum_{p \equiv a(m)} \frac{h}{p^s}$ must as well, implying there are infinitely many primes in this sum. $\square$

3.4. The prime number theorem

Theorem 3.9 (Hadamard-De La Vallée Poussin)
If $t \neq 0$, then $\zeta(1 + it) \neq 0$.

Proof. Let $s = \sigma + it$. If $\sigma > 1$, then

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \qquad \log \zeta(s) = \sum_{k,p} \frac{1}{kp^{ks}}.$$

Note that if $z = re^{i\theta}$,

$$\log|z| = \log|r| + \log|e^{i\theta}| = \log|r| = \operatorname{Re}(\log z).$$

Hence,

$$\log|\zeta(s)| = \operatorname{Re}(\log \zeta(s)).$$

Set

$$a_m := \begin{cases} \frac{1}{m} & \text{if } n = p^m, \\ 0 & \text{otherwise,} \end{cases}$$

so $\log \zeta(s) = \sum_n a_n n^{-s}$. Since $n^{-s} = e^{-(\log n) \cdot s}$,

$$\operatorname{Re}(\log \zeta(s)) = \sum_n a_n \cos((\log n) \cdot t) n^{-s}.$$

Now note the identity

$$\begin{aligned} 0 &\leq 2(1 + \cos\theta)^2 \\ &= 2 + 4\cos\theta + 2\cos^2\theta \\ &= 2 + 4\cos\theta + \cos^2\theta + \sin^2\theta + \cos^2\theta - \sin^2\theta \\ &= 3 + 4\cos\theta + \cos 2\theta, \end{aligned}$$

so

$$\begin{aligned}\log|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| &= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\ &= \sum_n a_n \underbrace{(3+4\cos((\log n)\cdot t) + \cos((2\log n)\cdot t))}_{\geq 0} n^{-\sigma} \\ &\geq 0.\end{aligned}$$

Therefore,

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 1.$$

We now send $\sigma \to 1^+$. If $\zeta(1+it) = 0$, then the LHS above tends to 0 as $\sigma \to 1$, since $\zeta^3(\sigma)$ has a pole of order 3 at $\sigma = 1$, which is a contradiction. □

Theorem 3.10 (Ikehara-Weiner)
Let $A(x)\colon [0,\infty) \to [0,\infty)$ be a non-decreasing function, and let

$$f(s) = \int_0^\infty A(x)e^{-xs}\,dx$$

and suppose it converges for $\sigma > 1$. Suppose furthermore that f admits a holomorphic extension to $U \supseteq \{s : \sigma \geq 1\}$ except possibly a simple pole at $s = 1$ with residue 1. Then

$$\lim_{x\to\infty} e^{-x}A(x) = 1.$$

That is, $A(x) \sim e^{-x}$.

Remark 3.11. This is an example of a Tauberian theorem. The idea here is that if I is the integral operator

$$A(x) \mapsto \int_0^\infty A(x)e^{-xs}\,dx,$$

then knowing information about $I\{A\}$ gives information about A.

Theorem 3.12
The following are equivalent:

(a) $\zeta(1+it) \neq 0$ for $t \in \mathbb{R}^\times$,

(b) $\lim_{x\to\infty} \psi(x)/x = 1$,

(c) $\lim_{x\to\infty} \pi(x)\log x/x = 1$,

(d) $\lim_{n\to\infty} p_n/n\log n = 1$, where p_n denotes the nth prime.

Proof. ((b) $\iff$ (c) (sketch)) Follows from the reasoning given in Lemma 1.30. For example, ((c) $\implies$ (b)) follows from the proof because it shows that

$$\begin{aligned} L_3 \le L_1 = 1, &\qquad L_3 \ge \alpha L_1 = \alpha \\ \ell_3 \le \ell_1 = 1, &\qquad \ell_3 \ge \alpha \ell_1 = \alpha, \end{aligned}$$

for $\alpha \in (0, 1)$, so taking $\alpha \to 1$ we get $L_3 = \ell_3 = 1$.

((c) $\implies$ (d)) We proved this on the homework. The problem is as follows:

Exercise 3.1. Let p_n denote the nth prime.

(a) Show that $\pi(p_n) = n$.

(b) Using $\pi(x) \sim \frac{x}{\log x}$, show that

$$n \sim \frac{p_n}{\log p_n}.$$

(c) Deduce that

$$p_n \sim n \log n.$$

[Hint: first show that $\log n \sim \log p_n$.]

((a) $\implies$ (b)) We will want to apply Ikehara-Weiner (3.10). Let $A(x) = \psi(e^x) = \sum_{n \le e^x} \Lambda(n)$. Then

$$\begin{aligned} f(s) &= \int_0^\infty A(x) e^{-xs}\, dx \\ &= \int_0^\infty \psi(e^x) e^{-xs}\, dx && (u = e^x) \\ &= \int_1^\infty \frac{\psi(u)}{u^{s+1}}\, du \\ &= -\frac{\zeta'(s)}{s\zeta(s)}, \end{aligned}$$

is meromorphic on $\sigma > 0$.

Recall from complex analysis that if m is an integer and $f \colon \mathbb{C} \to \mathbb{C}$ is a function that can be written as

$$f(s) = (s - s_0)^m g(s),$$

where $g(s_0) \neq 0$, then its log derivative can be written as

$$\frac{f'(s)}{f(s)} = \frac{m}{s - s_0} + \frac{g'(s)}{g(s)}.$$

Therefore, the log derivative has a pole at s_0 if and only if f has a root or pole at s_0.

Since $\zeta(s)$ has no roots on $\{1 + it : t \in \mathbb{R}^\times\}$ by assumption, $-\zeta'(s)/s\zeta(s)$ is holomorphic on this line.

By Theorem 3.10, $A(x) = \psi(e^x) \sim e^x$, which implies that $\psi(x) \sim x$.

((b) $\implies$ (a)) Assume that $\psi(x) \sim x$. Then

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx.$$

Let

$$\Phi(s) = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\psi(x)-x}{x^{s+1}}\,dx.$$

For $\varepsilon > 0$, let $x_0 \in \mathbb{R}$ be such that $x \geq x_0$ implies $|\psi(x)/x - 1| < \varepsilon \implies |\psi(x) - x| < \varepsilon|x|$. Then for $\sigma > 1$,

$$\begin{aligned}
|\Phi(s)| &= \left|\int_1^\infty \frac{\psi(x)-x}{x^{s+1}}\,dx\right| \\
&\leq \int_1^\infty \left|\frac{\psi(x)-x}{x^{s+1}}\right| dx \\
&= \int_1^{x_0} \frac{|\psi(x)-x|}{x^2}\,dx + \int_{x_0}^\infty \frac{\varepsilon}{x^\sigma}\,dx \\
&< K + \frac{\varepsilon}{\sigma-1},
\end{aligned}$$

for some constant $K > 0$. Hence,

$$|(\sigma-1)\Phi(\sigma+it)| < K(\sigma-1)+\varepsilon,$$

which implies

$$\lim_{\sigma\to 1} |(\sigma-1)\Phi(\sigma+it)| = 0.$$

Since $\sigma - 1 = (\sigma+it) - (1+it)$, $\zeta(s)$ has no zero or pole at $1+it$.

((d) $\implies$ (c)) Let (x_n) be a sequence such that $p_n \leq x_n < p_{n+1}$. Then $\pi(x_n) = n$. Since $p_n \sim n\log n$, $p_{n+1} \sim (n+1)\log(n+1) = n\log n$. Therefore, $x_n \sim n\log n$, so $x_n \sim \pi(x_n)\log\pi(x_n)$.

Claim 3.1. $\log\pi(x_n) \sim \log x_n$.

Proof. We use the fact that if $a_n \sim b_n$ are sequences going to infinity, then

$$\log b_n = \log a_n + \underbrace{\log\left(\frac{b_n}{a_n}\right)}_{\xrightarrow{n\to\infty}\log 1 = 0},$$

so $\log b_n \sim \log a_n$. Since $x_n \sim \pi(x_n)\log\pi(x_n)$,

$$\log x_n \sim \log\pi(x_n) + \log\log\pi(x_n) \sim \log\pi(x_n),$$

where the last "$\sim$" is justified by the fact that $\log\pi(x_n) \xrightarrow{n\to\infty} \infty$. ■

Hence, $x_n \sim \pi(x_n)\log x_n \implies \pi(x_n) \sim x_n/\log x_n$.

We now prove that this implies $\pi(x) \sim x/\log x$. By Chebyshev's theorem (1.31), $\frac{\pi(x)}{x/\log x}$ converges. If $(x_n')_n$ is any sequence that tends to ∞, choose some (increasing) subsequence $(x'_{\sigma(n)})_n$ such that, for each $m \geq 1$, $[p_m, p_{m+1})$ contains at most one $x'_{\sigma(n)}$. Then $(x'_{\sigma(n)})$ is some subsequence of some (x_n) satisfying $p_n \leq x_n < p_{n+1}$. Because, $\pi(x_n) \sim x_n/\log x_n$, we have

$$\frac{\pi(x'_{\sigma(n)})}{x_{\sigma(n)'}\log x_{\sigma(n)'}} \xrightarrow{n\to\infty} 1.$$

But since $\frac{\pi(x)}{x/\log x}$ converges, and we have proved a subsequence converges to 1, it must also converge to 1, meaning $\pi(x) \sim x/\log x$. $\square$

A. Some Fourier analysis

March 13, 2026 The main point of this section is to prove the Ikehara-Weiner theorem (3.10) and give some prerequisites to prove the analytic continuation of $\zeta(s)$ to $\mathbb{C}$, which we prove in the next section.

A.1. Pre-measure theory Fourier analysis

There are various notions of the Fourier transform:

Example A.1 (Fourier transforms) –

- If $f\colon \mathbb{R} \to \mathbb{C}$, then we define

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,dx.$$

- If $f\colon \mathbb{R} \to \mathbb{C}$ is periodic, that is, $f(x+1) = f(x)$ for all $x \in \mathbb{R}$, then we define

$$\widehat{f}(n) := \int_{0}^{1} f(x)e^{-2\pi inx}\,dx.$$

  Since f is periodic, it makes sense to talk about f as a function on $S^1 = \mathbb{R}/\mathbb{Z}$. If you do know measure theory, we can give S^1 the pushforward measure under the map $[0,1) \to S^1\colon t \mapsto e^{2\pi it}$, where $[0,1)$ has the standard Lebesgue measure.

- If $f\colon \mathbb{Z} \to \mathbb{C}$, then we define

$$\widehat{f}(\theta) := \sum_{n\in\mathbb{Z}} f(n)e^{2\pi in\theta}.$$

- (Not exactly a Fourier transform but still related) If $f\colon \mathbb{R}_{>0} \to \mathbb{C}$, then the Mellin transform is given by

$$\widehat{f}(\xi) := \int_{0}^{\infty} f(x)x^{-i\xi}\,\frac{dx}{x}.$$

Remark A.1. The general trend is that the groups $\mathbb{R}$, S^1, and $\mathbb{Z}$ are all *locally compact abelian (LCA)* groups. For any LCA, G, we can consider the continuous homomorphisms

$$\xi\colon G \to \mathbb{R}/\mathbb{Z}.$$

These form a group, which we call the *Pontryagin dual* of G, denoted $\widehat{G}$.

For example, we can show that $\widehat{\mathbb{R}} \cong \mathbb{R}$, $\widehat{\mathbb{R}/\mathbb{Z}} \cong S^1$, and $\widehat{S^1} \cong \mathbb{R}/\mathbb{Z}$. As the word dual suggests, $\widehat{\widehat{G}} = G$ naturally. This is known as *Pontryagin duality*.

A.1.1. Fourier analysis on $\mathbb{R}$

We start with Fourier analysis on functions $f\colon \mathbb{R} \to \mathbb{C}$. For this subsection (though not for the next subsection) we say a function f is in $L^1(\mathbb{R})$ if it is integrable and

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

We define the **Fourier transform** of f at $\xi \in \mathbb{R}$ as

$$\mathfrak{F}f(\xi) \coloneqq \widehat{f}(\xi) \coloneqq \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx.$$

Since $f \in L^1$ and $|e^{-2\pi i x\xi}| = 1$ for all $x, \xi \in \mathbb{R}$, the above integral converges for all $\xi \in \mathbb{R}$.

Lemma A.2 (Fourier transforms decay quickly)
If $f\colon \mathbb{R} \to \mathbb{C}$ is smooth and $f^{(n)}$ is in $L^1(\mathbb{R})$ for all n, then

$$\begin{aligned} |\widehat{f}(\xi)| &\le |2\pi\xi|^{-n} \left\| f^{(n)} \right\|_1 \\ &= O\left(|\xi|^{-n}\right). \end{aligned}$$

Proof. For all n, $f^{(n)} \in L^1$, so

$$f^{(n)}(x) - f^{(n)}(0) = \int_0^x f^{(n+1)}(t)\, dt \xrightarrow{x\to\infty} \int_0^\infty f^{(n+1)}(t)\, dt.$$

Therefore, $f^{(n)}(x)$ converges as $x \to \infty$. But $f^{(n)}$ is in L^1 only if $f^{(n)}(x) \to 0$ as $|x| \to \infty$. Therefore, by repeated applications of integration by parts,

$$\begin{aligned} \widehat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx \\ &= \left[\frac{f(x)e^{-2\pi i x\xi}}{2\pi i x\xi}\right]_{\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{f'(x)e^{-2\pi i x\xi}}{2\pi i\xi}\, dx = -\int_{-\infty}^{\infty} \frac{f'(x)e^{-2\pi i x\xi}}{2\pi i\xi}\, dx \\ &\;\;\vdots \\ &= (-1)^n \int_{-\infty}^{\infty} \frac{f^{(n)}(x)e^{-2\pi i x\xi}}{(2\pi i\xi)^n}\, dx. \end{aligned}$$

Taking the absolute values of both sides and noting that $|e^{-\pi i n x\xi}| = 1$, we get the result. □

This motivates the following definition:

Definition A.1

A function $f\colon \mathbb{R} \to \mathbb{C}$ is **Schwarz** if

$$\partial^n f(x) = O(x^{-m})$$

for all $n, m \geq 0$. We denote the space of such functions by $\mathcal{S}(\mathbb{R})$.

Similarly, the space of all 1-period smooth functions is denoted $\mathcal{S}(\mathbb{R}/\mathbb{Z})$, and the space of all functions $f\colon \mathbb{Z} \to \mathbb{C}$ such that $f(n) = O(|n|^{-k})$ for all $k \geq 0$ as $\mathcal{S}(\mathbb{Z})$.

Lemma A.3

If $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$.

Proof. We have that

$$\begin{aligned}
\frac{\widehat{f}(\xi+\varepsilon) - \widehat{f}(\xi)}{\varepsilon} &= \int_{-\infty}^{\infty} f(x) \left(\frac{e^{-2\pi i x \varepsilon} - 1}{\varepsilon} \right) e^{-2\pi i x \xi} \, dx \\
&= \int_{-\infty}^{\infty} f(x) \left(-2\pi i x + O(x^2 \varepsilon) \right) e^{-2\pi i x \xi} \, dx \\
&= \int_{-\infty}^{\infty} -2\pi i x f(x) e^{-2\pi i x \xi} \, dx + \int_{-\infty}^{\infty} O(x^2\varepsilon) f(x) e^{-2\pi i x \xi} \, dx.
\end{aligned}$$

Let $\varepsilon \to 0$. Then since f is a Schwarz function, the second integral tends to 0, and the left integral tends to $\mathfrak{F}(-2\pi i x f(x))(\xi)$. Since $-2\pi i x f(x) \in \mathcal{S}(\mathbb{R})$, $\widehat{f}'(\xi) = \mathfrak{F}(-2\pi i x f(x))$ converges everywhere. By Lemma A.2, $\widehat{f}'(\xi) = O(|\xi|^{-n})$ for all n. Repeating the procedure above with $\widehat{f}'$ shows that the second derivative exists and satisfies $\widehat{f}^{(2)}(\xi) = O(|\xi|^{-n})$ for all n. By induction, $\widehat{f} \in \mathcal{S}(\mathbb{R})$. $\square$

A.1.2. Fourier inversion

Lemma A.4

March 18, 2026

If $f, g \in \mathcal{S}(\mathbb{R})$, then

$$\int_{\mathbb{R}} \widehat{f} g = \int_{\mathbb{R}} f \widehat{g}.$$

Proof.

$$\begin{aligned}\int_{\mathbb{R}} \widehat{f}(x)g(x)\,dx &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y)e^{-2\pi i y x}\,dy\right) g(x)\,dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y)g(x)e^{-2\pi i y x}\,dy\right) dx \\ &= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x)e^{-2\pi i y x}\,dx\right) dy \\ &= \int_{\mathbb{R}} f(y)\widehat{g}(y)\,dy.\end{aligned}$$

The only part we need to justify is applying Fubini to swap the integrals from the second to third line. We can do this because

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)g(y)e^{-2\pi i y x}|\,dy\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)|\cdot|g(y)|\,dy\,dx \\ &= \int_{\mathbb{R}} |f(x)|\,dx \cdot \int_{\mathbb{R}} |g(y)|\,dy \\ &= \|f\|_1 \cdot \|g\|_1\,.\end{aligned}$$

□

One particular Fourier transform will be particularly important to the analytic continuation of ζ to $\mathbb{C}$ and to proving Fourier inversion: the Fourier transform of the Gaussian.

Lemma A.5 (Fourier transform of the Gaussian)

Suppose $t \in \mathbb{R}_{>0}$, and set

$$f_t(x) := e^{-\pi x^2 t}.$$

Then

$$\widehat{f}_t(\xi) := e^{-\pi\xi^2/t}/\sqrt{t}.$$

Proof. We have the relation

$$\begin{aligned}\widehat{f}_t(\xi) &= \int_{-\infty}^{\infty} e^{-\pi x^2 t - 2\pi i x\xi}\,dx \\ &= \int_{-\infty}^{\infty} e^{-\pi u^2 - 2\pi i u\left(\frac{\xi}{\sqrt{t}}\right)}\,\frac{du}{\sqrt{t}} && (u = x\sqrt{t}) \\ &= \widehat{f}_1(\xi/\sqrt{t})/\sqrt{t}.\end{aligned}$$

Therefore, it suffices to prove that

$$\widehat{f}_1(\xi) = e^{-\pi\xi^2}.$$

We will use complex integration over a certain contour. To figure out what function to integrate over, note that

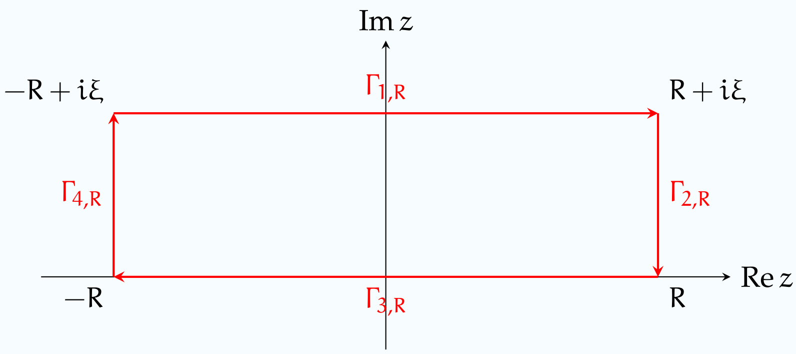
$$x^2 + 2ix\xi = (x + i\xi)^2 + \xi^2.$$

Therefore,

$$\widehat{f_1}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x\xi}\, dx = e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\, dx.$$

Our goal now is to compute the blue integral.

Consider the contour below:



Let $f(z) = e^{-\pi z^2}$. Then the complex integral of f along $\Gamma_{1,R}$ is

$$\int_{\Gamma_{1,R}} f(z)\, dz = \int_{-R}^{R} e^{-\pi(x+i\xi)^2}\, dx,$$

so

$$\lim_{R\to\infty} \int_{\Gamma_{1,R}} f(z)\, dz = \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\, dx.$$

Since f is holomorphic,

$$\left(\int_{\Gamma_{1,R}} + \int_{\Gamma_{2,R}} + \int_{\Gamma_{3,R}} + \int_{\Gamma_{4,R}} \right) f(z)\, dz = 0.$$

We first show there is no contribution from the integrals over $\Gamma_{2,R}$ and $\Gamma_{4,R}$. On $\Gamma_{2,R}$, for $t \in [0, \xi]$,

$$f(z) = e^{-\pi(R+it)^2} \implies |f(z)| = e^{-\pi(R^2-t^2)} \leq e^{-\pi(R^2-\xi^2)}.$$

Since $\ell(\Gamma_{2,R}) = \xi$ is constant, and $e^{-\pi(R^2-\xi^2)} \xrightarrow{R\to\infty} 0$, there is no contribution from the integral over $\Gamma_{2,R}$. The same reasoning words for $\Gamma_{4,R}$. Therefore,

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-\pi(x+i\xi)^2}\, dx = \lim_{R\to\infty} \int_{-R}^{R} e^{-\pi x^2}\, dx = 1.$$

Therefore, $\widehat{f_1}(\xi) = e^{-\pi\xi^2}$. $\square$

For $\varepsilon > 0$, define

$$g_\varepsilon(t) := e^{-\pi t^2/\varepsilon^2}/\varepsilon,$$
$$\phi_\varepsilon(t) := e^{-\pi t^2 \varepsilon^2}.$$

By Lemma A.5, $g_\varepsilon(t) = \widehat{\phi_\varepsilon}(t)$. Also note that

$$\int_{-\infty}^{\infty} \phi_\varepsilon(t)\, dt = 1.$$

But, looking at graphs of the function g_ε as $\varepsilon \to \infty$, we notice that it "spikes" more at $t = 0$. Therefore, we should view g_ε as approximations of the Dirac delta function at $t = 0$. We'll want to show, for any $f \in \mathcal{S}(\mathbb{R})$,

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} g_\varepsilon(t) f(t)\, dt = f(0). \tag{A.1}$$

On the other hand, ϕ_ε seems to "spread out" as $\varepsilon \to 0$, and begins to look like a constant function that is 1 everywhere, so we would hope that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \phi_\varepsilon(t) f(t)\, dt = \int_{-\infty}^{\infty} f(t)\, dt \tag{A.2}$$

for any $f \in \mathcal{S}(\mathbb{R})$.

Theorem A.6 (Fourier inversion)
Suppose $f \in \mathcal{S}(\mathbb{R})$. Then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi}\, d\xi.$$

Proof. We claim it suffices to prove this result at $x = 0$. Indeed suppose it holds for $x = 0$ and let T_a be the operator sending $f(x)$ to $f(x + a)$. Then

$$\begin{aligned}
\widehat{T_a f}(\xi) &= \int_{-\infty}^{\infty} f(x + a) e^{-2\pi i x \xi}\, dx \\
&= e^{2\pi i \xi a} \int_{-\infty}^{\infty} f(u) e^{-2\pi i u \xi}\, du && (u = x + a) \\
&= e^{2\pi i \xi a} \widehat{f}(\xi).
\end{aligned}$$

Therefore,

$$f(a) = T_a f(0) = \int_{-\infty}^{\infty} \widehat{T_a f}(\xi)\, d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i \xi a}\, d\xi.$$

Therefore, we show that

$$f(0) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, d\xi.$$

If we prove the equations (A.1), (A.2) above, then

$$\begin{aligned}
f(0) &\overset{(A.1)}{=} \lim_{\varepsilon\to\infty}\int_{-\infty}^{\infty} g_\varepsilon(t)f(t)\,dt \\
&= \lim_{\varepsilon\to\infty}\int_{-\infty}^{\infty} \widehat{\phi}_\varepsilon(t)f(t)\,dt \\
&\overset{(A.4)}{=} \lim_{\varepsilon\to\infty}\int_{-\infty}^{\infty} \phi_\varepsilon(t)\widehat{f}(t)\,dt \\
&\overset{(A.2)}{=} \int_{-\infty}^{\infty} \widehat{f}(t)\,dt.
\end{aligned}$$

We prove these equations with basic analysis.

[TODO: finish the proof] □

Proposition A.7 (Plancherel's formula)
Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\,d\xi.$$

Proof. Consider $g = \overline{\widehat{f}}$. Then

$$\int_{\mathbb{R}} f\widehat{g} = \int_{\mathbb{R}} \widehat{f}g = \int_{\mathbb{R}} |\widehat{f}|^2.$$

So it suffices to compute the LHS. By Fourier inversion,

$$\widehat{g}(x) = \int_{-\infty}^{\infty} \overline{\widehat{f}}(\xi)e^{-2\pi i x\xi}\,d\xi = \overline{\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi} \overset{(A.6)}{=} \overline{f(x)}.$$ □

A.1.3. Fourier analysis on $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}$

Definition A.2
Let $f \in \mathcal{S}(\mathbb{Z})$. Define

$$\widehat{f}(\theta) := \sum_{n\in\mathbb{Z}} f(n)e^{-2\pi i n\theta}.$$

Let $g \in \mathcal{S}(\mathbb{R}/\mathbb{Z})$. Define

$$\widehat{g}(n) := \int_0^1 f(\theta)e^{-2\pi i\theta}\,d\theta.$$

It turns out $\widehat{f} \in \mathcal{S}(\mathbb{R}/\mathbb{Z})$[1] and $\widehat{g} \in \mathcal{S}(\mathbb{Z})$. Inversion still holds:

[1]This is a fairly trivial statement, all that needs to be checked is that $\widehat{f}$ is smooth.

Theorem A.8 (Fourier inversion for $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}$)
If $f \in \mathcal{S}(\mathbb{Z})$ and $g \in \mathcal{S}(\mathbb{R}/\mathbb{Z})$, then

$$f(n) = \int_0^1 \widehat{f}(\theta)e^{2\pi i n\theta}\, d\theta,$$
$$g(\theta) = \sum_{n\in\mathbb{Z}} \widehat{g}(n)e^{2\pi i n\theta}.$$

Theorem A.9 (Poisson summation)
Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m).$$

Proof. Let $F, G \colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be defined by

$$F(\theta) := \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi i n\theta},$$
$$G(\theta) := \sum_{m\in\mathbb{Z}} f(\theta + m).$$

The mth Fourier coefficent of F is

$$\begin{aligned}
\widehat{F}(m) &= \int_0^1 F(\theta)e^{-2\pi i m\theta}\, d\theta \\
&= \int_0^1 \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi i\theta(n-m)}\, d\theta \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 \widehat{f}(n)e^{2\pi i\theta(n-m)}\, d\theta \\
&= \widehat{f}(m).
\end{aligned}$$

The mth Fourier coefficient of G is

$$\begin{aligned}
\widehat{G}(m) &= \int_0^1 \left(\sum_{n\in\mathbb{Z}} f(\theta+n)e^{-2\pi i m\theta} \right) d\theta \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 f(\theta+n)e^{-2\pi i m\theta}\, d\theta \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(u)e^{-2\pi i m(u-n)}\, du && (u=\theta+n) \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(u)e^{-2\pi i m u}\, du && \text{(periodicity)} \\
&= \int_{-\infty}^{\infty} f(u)e^{-2\pi i m u}\, du \\
&= \widehat{f}(m).
\end{aligned}$$

By Fourier inversion (A.8), F and G are the same when evaluated at $\theta = 0$. □

A.2. Fourier analysis with measure theory

March 25, 2026 The above results are enough to finish the proof of the analytic continuation of $\zeta(s)$ to $\mathbb{C}$. We now focus on Fourier analysis with *Lebesgue measurable functions*. My notes on analysis or [Fol99, Ch. 1-3] give an introduction to Lebesgue integration. Moreover, [Fol99, Ch. 8] gives a more in-depth treatment of Fourier analysis.

We let $L^1(\mathbb{R})$ be the space of (Lebesgue) measurable functions $f\colon \mathbb{R} \to \mathbb{C}$ with $\int_{\mathbb{R}} |f(x)|\, dx < \infty$ (modulo a.e. equivalence). This space has the norm $\|f\|_1 := \int_{\mathbb{R}} |f|$. The most important result is the *dominated convergence theorem*:

Theorem A.10 (Dominated convergence theorem)
If $(f_n\colon \mathbb{R} \to \mathbb{C})$ are a series of functions in $L^1(\mathbb{R})$ that converge almost everywhere (a.e.) to a function f, and there exists a function $g \in L^1$ such that $|f_n| \le g$ a.e., then $f \in L^1$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f.$$

Another result we need is

Proposition A.11
The space of compactly supported, continuous function $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.

Proof (sketch). By separating real and imaginary parts, it suffices to consider real-valued functions. By separating positive and negative parts, it suffices to consider positive functions. By the construction of the Lebesgue integral, it suffices to consider indicator functions on E, where E is measurable.

There is a result that, for any $\varepsilon > 0$, there exists an open set U and a compact set K such that $K \subseteq E \subseteq U$, and $|U - K| < \varepsilon$. Now apply Urysohn's lemma. $\square$

For $a \in \mathbb{R}$, let T_a be the operator sending $f(x)$ to $f(x+a)$. Note that $\|f\|_1 = \|T_a f\|_1$ by the substitution $u = x + a$.

Lemma A.12

The action $\mathbb{R} \curvearrowright L^1(\mathbb{R})$ defined by $a \cdot f := T_a f$ is continuous.

Since any action $G \curvearrowright X$ can be written as a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x, \end{aligned}$$

we say that this action is *continuous* if the above map is continuous.[2]

Proof. Note that is suffices to check continuity at $0 \in \mathbb{R}$ because

$$\|T_a f - T_a g\|_1 = \|f - g\|_1 .$$

Fix an $\varepsilon > 0$. Since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, choose a compactly supported g with support in $[-K, K]$, $K > 1$, so that $\|f - g\| < \varepsilon$. By uniform continuity, there exists a $\delta \in (0, 1)$ so that $|x - y| < \delta$ implies

$$|g(x) - g(y)| < \frac{\varepsilon}{4K}.$$

Then for $|a| < \delta$,

$$\|T_0 g - T_a g\|_1 = \|g - T_a g\|_1 = \int_{\mathbb{R}} |g(x) - g(x+a)|\, dx < 2(K + \delta)\frac{\varepsilon}{4K} \leq \varepsilon. \quad \square$$

Example A.2 – Let $f = \chi_{[a,b]}$ be the indicator function on $[a, b]$. Then

$$\begin{aligned} \widehat{f}(\xi) &= \int_a^b e^{-2\pi i x \xi}\, dx \\ &= \begin{cases} \frac{e^{-2\pi i b \xi} - e^{-2\pi i a \xi}}{-2\pi i \xi} & \text{if } \xi \neq 0, \\ b - a & \text{if } \xi = 0. \end{cases} \end{aligned}$$

One can check that $\widehat{f}$ is *not* discontinuous at $\xi = 0$, so, while f is certainly not continuous, $\widehat{f}$ is.

[2] Note that G must necessarily have a topological structure to be talking about continuity!

Theorem A.13 (The Riemann-Lebesgue lemma)
If $f \in L^1(\mathbb{R})$, then

(a) $\widehat{f}$ is continuous,

(b) $\widehat{f}(\xi) \to 0$ as $\xi \to \infty$,

(c) $\|\widehat{f}\|_\infty \leq \|f\|_1$.

If we let $C_0(\mathbb{R})$ be the space of real-valued continuous functions that vanish as $|x| \to \infty$, then the above theorem says that $\mathfrak{F}L^1(\mathbb{R}) \subseteq C_0(\mathbb{R})$.

Proof of Theorem A.13. (c)

$$\begin{aligned}
\|\widehat{f}\|_\infty &= \sup_\xi \left|\widehat{f}(\xi)\right| \\
&= \sup_\xi \left|\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,dx\right| \\
&\leq \sup_\xi \int_{\mathbb{R}} |f(x)|\,dx \\
&= \|f\|_1 .
\end{aligned}$$

(a) Let (ξ_n) be a sequence converging to ξ. Then

$$\begin{aligned}
\left|\widehat{f}(\xi_n) - \widehat{f}(\xi)\right| &\leq \left|\int_{\mathbb{R}} f(x)\left[e^{-2\pi i x\xi_n} - e^{-2\pi i x\xi}\right]dx\right| \\
&\leq \int_{\mathbb{R}} |f(x)| \cdot 2\,dx \\
&= 2\,\|f\|_1 .
\end{aligned}$$

By the dominated convergence theorem (A.10),

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\left[e^{-2\pi i x\xi_n} - e^{-2\pi i x\xi}\right]dx = \int_{\mathbb{R}} 0\,dx = 0.$$

(b) Instead we consider $-\widehat{f}(\xi)$:

$$\begin{aligned}
-\widehat{f}(\xi) &= -\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,dx \\
&= e^{i\pi}\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,dx \\
&= \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi + i\pi}\,dx.
\end{aligned}$$

By the substitution $u = x - \frac{1}{2\xi}$,

$$\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi + i\pi}\, dx = \int_{\mathbb{R}} f\left(u + \frac{1}{2\xi}\right) e^{-2\pi i u\xi}\, du$$
$$= \int_{\mathbb{R}} (T_{1/2\xi} f)(u)e^{-\pi i u\xi}\, du.$$

Hence,

$$\widehat{f}(\xi) = \frac{\widehat{f}(\xi) - (-\widehat{f}(\xi))}{2} = \frac{1}{2}\int_{\mathbb{R}} (f - T_{1/2\xi} f)e^{-2\pi i x\xi}\, dx.$$

Therefore,

$$\left|\widehat{f}(\xi)\right| \leq \frac{1}{2}\left\|f - T_{1/2\xi} f\right\|_1 .$$

By Lemma A.12, the limit as $\xi \to \infty$ of the RHS is zero. □

Proposition A.14

Let $f \in L^1(\mathbb{R})$ and $\zeta \in \mathbb{R}$. Then

1. $\widehat{T_a f}(\zeta) = e^{2\pi i\xi a}\widehat{f}(\xi)$.
2. Let $M_a\{f\}(x) \coloneqq e^{2\pi i x a} f(x)$. Then

$$\widehat{M_a\{f\}}(\xi) = \widehat{f}(\xi - a).$$

Definition A.3

Let $f, g \in L^1(\mathbb{R})$. The **convolution** of f and g is defined as

$$(f * g)(x) \coloneqq \int_{\mathbb{R}} f(x-y)g(y)\, dy.$$

We have that $f * g \in L^1(\mathbb{R})$ because

$$\begin{aligned}
\|f * g\|_1 &= \int_{\mathbb{R}} \left|\int_{\mathbb{R}} f(x-y)g(y)\, dy\right| dx \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)g(y)|\, dy\, dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-y)|\, dx\right) |g(y)|\, dy \\
&= \int_{\mathbb{R}} \|f\|_1 |g(y)|\, dy \\
&= \|f\|_1 \cdot \|g\|_1 < \infty.
\end{aligned}$$

This makes the map $- * -: L^1(\mathbb{R}) \times L^1(\mathbb{R}) \to L^1(\mathbb{R})$ associative, commutative, and $\mathbb{C}$-bilinear. However, it doesn't have an identity, which we will show now.

The following proposition is proven by writing out the definitions.

Proposition A.15
If $f, g \in L^1$, then $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.

Corollary A.16
$L^1(\mathbb{R})$ does not have an identity element under the operation $*$.

Proof. Suppose $f * g = f$, so $\widehat{f} \cdot \widehat{g} = \widehat{f}$. If we take $f = \chi_{[0,b]}$ for some $b > 0$, then

$$\widehat{f}(\xi) = \begin{cases} \frac{e^{-2\pi i b \xi} - 1}{-2\pi i \xi} & \text{if } \xi \neq 0, \\ b & \text{if } \xi = 0. \end{cases}$$

Therefore, $\widehat{f}(\xi) = 0 \iff b\xi \in \mathbb{Z}$. But then $\widehat{g}(\xi) = 1$ whenever $b\xi \notin \mathbb{Z}$, which implies that $\widehat{g}(\xi) \not\to 0$ as $\xi \to \infty$. □

A.3. Laplace transform

Definition A.4
For $f \in L^1$, we define the **Laplace transform** of f at $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$ as

$$\mathcal{L}\{f\}(s) := \int_0^\infty f(x) e^{-xs}\, dx.$$

Remark A.17. Notice that

$$\mathcal{L}\{f\}(x + iy) = \int_0^\infty f(t) e^{-xt} e^{-iyt}\, dt,$$

which is *nearly* $\mathfrak{F}\{f(t)e^{-xt}\}$, except the bounds of integration aren't on $\mathbb{R}$.

For $A < 0$, if $f(t)e^{-At} \in L^1(\mathbb{R})$, then the Laplace transform is also well-defined on $\operatorname{Re} s \geq A$.

Example A.3 – For

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ e^{-Ax} & \text{if } x \geq 0, \end{cases}$$

we have

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \int_0^\infty e^{-(A+s)x}\,dx \\ &= \left[\frac{e^{-(A+s)x}}{-(A+s)}\right]_0^\infty \\ &= \begin{cases}\frac{1}{A+s} & \text{if } \operatorname{Re} s > -A, \\ \text{undefined} & \text{otherwise.}\end{cases}\end{aligned}$$

A.4. Proof of Weiner-Ikehara

We are now ready to prove Theorem 3.10.

Before we begin, we need to consider the following function:

$$\phi_\lambda(t) \coloneqq \left(1 - \frac{|t|}{2\lambda}\right)\chi_{[-2\lambda,2\lambda]},$$

whose graph looks like a triangle, as shown in Figure 4.

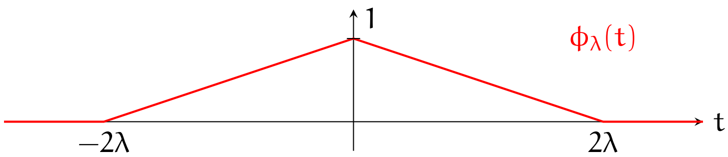


Figure 4: Graph of $\phi_\lambda(t)$. Made with help from GitHub Copilot.

We also consider the following integral transform of ϕ_λ:

$$h_\lambda(x) \coloneqq \int_{\mathbb{R}} \phi_\lambda(t)e^{itx}\,dt$$

Note that

$$\begin{aligned}
\int_{\mathbb{R}} \phi_\lambda(t)e^{itx}\,dt &= \int_{-2\lambda}^{0}\left(1+\frac{t}{2\lambda}\right)e^{itx}\,dt + \int_{0}^{2\lambda}\left(1-\frac{t}{2\lambda}\right)e^{itx}\,dt \\
&= \int_{-2\lambda}^{2\lambda} e^{itx}\,dt + \frac{1}{2\lambda}\left(\int_{-2\lambda}^{0} te^{itx}\,dt - \int_{0}^{2\lambda} te^{itx}\,dt\right) \\
&= \frac{e^{i2\lambda x}-e^{-i2\lambda x}}{ix} + \frac{1}{2\lambda}\left(-\frac{1}{x^2}+\frac{2\lambda e^{-ix2\lambda}}{ix}+\frac{e^{-ix2\lambda}}{x^2}-\frac{2\lambda e^{ix2\lambda}}{ix}+\frac{e^{ix2\lambda}}{x^2}-\frac{1}{x^2}\right) \\
&= \frac{1}{2\lambda}\cdot\frac{e^{ix2\lambda}+e^{-ix2\lambda}-2}{x^2} \\
&= \frac{2\cos(x2\lambda)-2}{2\lambda x^2} \\
&= \frac{\cos(x2\lambda)-1}{\lambda x^2} \\
&= \frac{2\sin^2 x\lambda}{\lambda x^2} \\
&= 2\lambda\left(\frac{\sin\lambda x}{\lambda x}\right)^2
\end{aligned}$$

If we let $B(x) = A(x)e^{-x}$, then the theorem is the same as saying that $B(x) - 1 \xrightarrow{x\to\infty} 0$. In theory, we would like to show that $B(x) - 1$ is a Fourier transform of some function, and apply the Riemann-Lebesgue lemma (A.13).

The first issue with this is that $B(x) - 1$ is not even defined on $x < 0$, which we fix by assuming that $B(x) = 0$ for $x < 0$ and considering $B - \chi_{[0,\infty)}$. Another issue is that B is not continuous, which we resolve by replacing $B - \chi_{[0,\infty)}$ with

$$(B - \chi_{[0,\infty)}) * h_\lambda.$$

We will show that the asymptotic behaviors of this function and $B - \chi_{[0,\infty)}$ are related.

Claim A.1.

$$\lim_{y\to\infty}(B * h_\lambda)(y) = 2\pi.$$

Proof. Since B is supported on $[0,\infty)$, we have that

$$\begin{aligned}
\lim_{y\to\infty}(B * h_\lambda)(y) &= \lim_{y\to\infty}\int_{-\infty}^{y} B(y-x)h_\lambda(x)\,dx \\
&= \lim_{y\to\infty}\int_{-\infty}^{y} B(y-x)2\lambda\left(\frac{\sin\lambda x}{\lambda x}\right)^2 dx.
\end{aligned}$$

Taking the substitution $v = \lambda x$, the limit becomes

$$2\lim_{y\to\infty}\int_{-\infty}^{\lambda y} B\left(y-\frac{v}{\lambda}\right)\frac{\sin^2 v}{v^2}\,dv,$$

so it is equivalent to prove that

$$\lim_{y\to\infty} \int_{-\infty}^{\lambda y} B\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv = \pi.$$

We have that

$$f(s) = \mathcal{L}\{A\}(s), \qquad \frac{1}{s-1} = \mathcal{L}\left\{e^{-x}\right\}(s).$$

Since f has a pole of order 1 at $s = 1$,

$$g(s) := f(s) - \frac{1}{s-1} = \mathcal{L}\left\{A(x) - e^{-x}\right\}(s) = \mathcal{L}\{B-1\}(s-1)$$

is a holomorphic function on $\sigma \geq 1$.

Let

$$g_\varepsilon(t) = g(1 + \varepsilon + it)$$

for $\varepsilon \geq 0$. For now, assume that $\varepsilon > 0$. We have that

$$\frac{1}{2}\int_{\mathbb{R}} g_\varepsilon(t) e^{iyt} \phi_\lambda(t)\, dt = \frac{1}{2}\int_{\mathbb{R}} \left(\int_0^\infty (B(x)-1)e^{-x(\varepsilon+it)}\, dx\right) e^{iyt}\phi_\lambda(t)\, dt.$$

We now wish to interchange the order of integration. Since $A(x)$ is non-negative and non-decreasing, for real σ and $x > 0$,

$$f(\sigma) = \int_0^\infty A(u)e^{-u\sigma}\, du \geq A(x)\int_x^\infty e^{-u\sigma}\, du = \frac{A(x)e^{-x\sigma}}{\sigma}.$$

Therefore, for $\sigma > 1$,

$$A(x) \leq \sigma f(\sigma) e^{x\sigma},$$

which implies

$$A(x) = O\left(e^{x\sigma}\right)$$

for $\sigma > 1$, so

$$A(x) = o\left(e^{x\sigma}\right)$$

for $\sigma > 1$. This implies that

$$B(x)e^{-\delta x} = A(x)e^{-(1+\delta)x} = o(1)$$

for $\delta > 0$, meaning the integral

$$\int_0^\infty (B(x)-1)e^{-(\varepsilon+it)x}\, dx$$

converges uniformly in t on $[-2\lambda, 2\lambda]$. Indeed,

$$\left|(B(x)-1)e^{-(\varepsilon+it)x}\right| \leq (B(x)+1)e^{-\varepsilon x} \leq Ce^{-(\varepsilon-\delta)x} + e^{-\varepsilon x},$$

where the constant C is because $B(x)e^{-\varepsilon x} = O(e^{-\delta x})$ for some $\delta \in (0, \varepsilon)$. The integral of the RHS from 0 to ∞ converges, and does not depend on t, so we have uniform convergence. This is enough to interchange the order of integration because the support of ϕ_λ is $[-2\lambda, 2\lambda]$. Therefore,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left(\int_0^\infty (B(x)-1)e^{-x(\varepsilon+it)}\,dx\right)e^{iyt}\phi_\lambda(t)\,dt \\
&= \int_0^\infty (B(x)-1)e^{-\varepsilon x}\left(\int_{\mathbb{R}}\frac{1}{2}e^{i(y-x)t}\phi_\lambda(t)\,dt\right)dx \\
&= \int_0^\infty (B(x)-1)e^{-\varepsilon x}\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx.
\end{aligned}$$

For $\sigma \geq 1$, $g(s)$ is analytic, so $g_\varepsilon(t) \to g(1+it)$ uniformly as $\varepsilon \to 0$ in t on the interval $[-2\lambda, 2\lambda]$. By the monotone convergence theorem,

$$\int_0^\infty e^{-\varepsilon x}\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx \xrightarrow{\varepsilon\to 0} \int_0^\infty \frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx.$$

The RHS clearly converges. Combining these two facts, the limit

$$\lim_{\varepsilon\to 0}\int_0^\infty B(x)e^{-\varepsilon x}\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx$$

exists. Again, by the monotone convergence theorem,

$$\lim_{\varepsilon\to 0}\int_0^\infty B(x)e^{-\varepsilon x}\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx = \int_0^\infty B(x)\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx.$$

Therefore,

$$\frac{1}{2}\int_{\mathbb{R}} g_\varepsilon(t)e^{iyt}\phi_\lambda(t)\,dt = \int_0^\infty B(x)\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx - \underbrace{\int_0^\infty \frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx}_{(1)}.$$

The LHS is a Fourier transform, so taking $y \to \infty$, it must converge to 0 by Theorem A.13. (1) satisfies

$$\lim_{y\to\infty}\int_0^\infty \frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}\,dx \overset{(\nu=\lambda(y-x))}{=} \lim_{y\to\infty}\int_{-\infty}^{\lambda y}\frac{\sin^2\nu}{\nu^2}\,dx = \pi.$$

Therefore,

$$\int_{-\infty}^{\lambda y} B\left(y - \frac{\nu}{\lambda}\right)\frac{\sin^2\nu}{\nu^2}\,d\nu = \pi. \qquad \blacksquare$$

Claim A.2. $\lim_{x\to\infty} B(x) = 1$.

Proof. We first show that $\limsup_{y\to\infty} B(y) \leq 1$ and then $\liminf_{y\to\infty} B(y) \geq 1$.

Let $a, \lambda > 0$ and choose a $y > a/\lambda$. By the previous claim, we have

$$\limsup_{y\to\infty} \int_{-a}^{a} B\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv \leq \pi,$$

since the integrand is non-negative. Since $A(x) = B(x)e^x$, $A(x)$ is non-decreasing, so for any $v \in [-a, a]$, we have

$$A\left(y - \frac{v}{\lambda}\right) \geq A\left(y - \frac{a}{\lambda}\right),$$

which implies

$$\begin{aligned} &B\left(y - \frac{v}{\lambda}\right) e^{y - \frac{v}{\lambda}} \geq B\left(y - \frac{a}{\lambda}\right) e^{y - \frac{a}{\lambda}} \\ &\implies B\left(y - \frac{v}{\lambda}\right) \geq B\left(y - \frac{a}{\lambda}\right) e^{\frac{(v-a)}{\lambda}} \geq B\left(y - \frac{a}{\lambda}\right) e^{-2a/\lambda}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\limsup_{y\to\infty} \int_{-a}^{a} B\left(y - \frac{a}{\lambda}\right) e^{-2a/\lambda} \frac{\sin^2 v}{v^2}\, dv \leq \pi \\ \implies &\limsup_{y\to\infty} B\left(y - \frac{a}{\lambda}\right) e^{-2a/\lambda} \int_{-a}^{a} \frac{\sin^2 v}{v^2}\, dv \leq \pi. \end{aligned}$$

Setting $\lambda = a^2$ and noting $a \leq y\lambda$, so $\frac{1}{a} \leq y$ and taking $a \to \infty$, we get that $\limsup_{y\to\infty} B(y) \leq 1$.

We prove $\liminf_{y\to\infty} B(y) \geq 1$ by using the first part. Since $\limsup_{y\to\infty} B(y) \leq 1$, $B(y)$ is bounded above by some c' for all $y \geq y_0$ for some some $y_0 > 0$. However, on $[0, y_0]$, we have

$$B(x) = A(x)e^{-x} \leq A(x) \leq A(y_0),$$

so we there exists some universal bound c of $|B(x)|$. Hence,

$$\begin{aligned} \int_{-\infty}^{\lambda y} B\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv \leq c \int_{|v|>a} \frac{\sin^2 v}{v^2}\, dv \\ + \int_{|v|\leq a} B\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv. \end{aligned}$$

As before, $B\left(y - \frac{v}{\lambda}\right) \leq B\left(y + \frac{a}{\lambda}\right) e^{2a/\lambda}$, which implies

$$\int_{|v|\leq a} B\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv \leq B\left(y + \frac{a}{\lambda}\right) e^{2a/\lambda} \int_{-a}^{a} \frac{\sin^2 v}{v^2}\, dv,$$

combining these facts,

$$\pi \leq \liminf_{y\to\infty}\left(c\int_{|v|>a}\frac{\sin^2 v}{v^2}\,dv + B\left(y+\frac{a}{\lambda}\right)e^{2a/\lambda}\int_{|v|\leq a}\frac{\sin^2 v}{v^2}\,dv\right).$$

Letting $\lambda = a^2$ and taking $a \to \infty$, we have that

$$\pi \leq \pi \liminf_{y\to\infty} B(y) \implies \liminf_{y\to\infty} B(y) \geq 1. \qquad \blacksquare$$

B. The functional equation for ζ and analytic continuation to $\mathbb{C}$

Recall that the **gamma function**, defined by

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t},$$

for $\sigma > 0$, is holomorphic. Note that $\Gamma(s)$ is the Mellin transform of e^{-t} at s.

To see that Γ is holomorphic, we use the following two lemmas, which are proven using Morera's theorem and Goursat's theorem.

Lemma B.1

Suppose $\{f_n\}$ is a sequence of holomorphic functions uniformly converging to a function f in every compact subset of a domain Ω. Then f is a holomorphic function on Ω.

Lemma B.2

Let $F(z, s)$ be a function on $\Omega \times [a, b]$, where Ω is a domain. Suppose

1. $F(z, s)$ is holomorphic in z for each s,
2. F is continuous.

Then the function f defined on Ω by

$$f(z) = \int_a^b F(z, s)\, ds$$

is holomorphic.

Theorem B.3

Γ is analytic on $\sigma > 0$.

Proof. Let

$$\Gamma_\varepsilon(s) := \int_\varepsilon^{1/\varepsilon} t^{s-1} e^{-t}\, dt. \tag{B.1}$$

By Lemma B.2, Γ_ε defines a holomorphic function. We now need to show that Γ_ε converges to Γ uniformly on compact subsets. We consider instead the strips $S_{\delta,M} := \{z : \delta < \mathrm{Re}(z) < M\}$ for $0 < \delta < M < \infty$. Then

$$|\Gamma(s) - \Gamma_\varepsilon(s)| \le \int_0^\varepsilon t^{\sigma-1} e^{-t}\, dt + \int_{1/\varepsilon}^\infty t^{\sigma-1} e^{-t}\, dt.$$

For the former integral, we have the bound

$$\left|\int_0^\varepsilon t^{\sigma-1}e^{-t}\,dt\right| \le \int_0^\varepsilon t^{\sigma-1}\,dt = \frac{\varepsilon^\delta}{\delta}.$$

For the latter integral, note that for sufficiently large t, $t^{M-1} \le e^{t/2}$. Therefore,

$$\left|\int_{1/\varepsilon}^\infty t^{\sigma-1}e^{-t}\,dt\right| \le \int_{1/\varepsilon}^\infty t^{M-1}e^{-t}\,dt \le \int_{1/\varepsilon}^\infty e^{-t/2}\,dt,$$

which is bounded and tends to 0 as $\varepsilon \to 0$. Therefore, the convergence is uniform.

Since we did this for arbitrary strips, Lemma B.1 implies that Γ is analytic on $\sigma > 0$. □

Lemma B.4

Γ has an analytic continuation to $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$. The residue of Γ at $s = -n$ is $(-1)^n/n!$.

Proof. We first prove that $\Gamma(s+1) = s\Gamma(s)$ for $\sigma > 0$. Integration by parts on Γ_ε (B.1) yields

$$e^{-1/\varepsilon}\left(\frac{1}{\varepsilon}\right)^s - e^\varepsilon\varepsilon^s = \int_\varepsilon^{1/\varepsilon}\frac{d}{dt}(e^{-t}t^s)\,dt = -\int_\varepsilon^{1/\varepsilon} e^{-t}t^s\,dt + s\int_\varepsilon^{1/\varepsilon} e^{-t}t^{s-1}\,dt.$$

As $\varepsilon \to 0$, the LHS tends to 0. Rearranging gives us $\Gamma(s+1) = s\Gamma(s)$.

To extend Γ, define, for $\sigma > -1$,

$$F_1(z) = \frac{\Gamma(s+1)}{s}.$$

This is defined for $\sigma > -1$, and for $\sigma > 0$,

$$F_1(z) = \frac{\Gamma(s+1)}{s} = \Gamma(s).$$

Similarly, define

$$F_m(z) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}$$

for $\sigma > -m$. The residue computations are easy from these analytic continuations. □

Now note that

$$\Gamma\left(\frac{1}{2}s\right) = \int_0^\infty t^{s/2-1}e^{-t}\,dt.$$

By substituting $t = n^2\pi x$, we have

$$\Gamma\left(\frac{1}{2}s\right) = \int_0^\infty n^2\pi\cdot(n^2\pi x)^{s/2-1}e^{-n^2\pi x}\,dx,$$

which implies

$$\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right) n^{-s} = \int_0^\infty x^{s/2-1} e^{-n^2\pi x}\, dt.$$

Taking the sum of these over all natural numbers, we get

$$\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right) \zeta(s) = \sum_{n=1}^{\infty} \int_0^\infty x^{s/2-1} e^{-n^2\pi x}\, dt.$$

Since

$$\begin{aligned}\sum_{n=1}^{\infty} \int_0^\infty \left| x^{s/2-1} e^{-n^2\pi x} \right| dx &\leq \sum_{n=1}^{\infty} \int_0^\infty x^{\sigma/2-1} e^{-n^2\pi x}\, dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2\pi}\Gamma\left(\frac{1}{2}s\right) < \infty,\end{aligned}$$

we have absolute convergence, hence we may exchange the sum and integral, so

$$\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right) \zeta(s) = \int_0^\infty x^{s/2-1} \sum_{n=1}^{\infty} e^{-n^2\pi x}\, dx.$$

Letting $\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$, we have

$$\begin{aligned}\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right) \zeta(s) &= \int_0^\infty x^{s/2-1}\omega(x)\, dx \\ &= \int_1^\infty x^{s/2-1}\omega(x)\, dx + \int_1^\infty x^{-s/2-1}\omega(x^{-1})\, dx.\end{aligned}$$

We introduce the function $\theta(x) = \sum_{i=-\infty}^{\infty} e^{-n^2\pi x}$, then $\theta(x) = 2\omega(x) + 1$.

Lemma B.5
For $x > 0$, θ satisfies the following functional equation: $\theta(x^{-1}) = x^{\frac{1}{2}}\theta(x)$.

Before we prove this, we show why it's useful. From this lemma it follows that

$$\omega(x^{-1}) = \frac{\theta(x^{-1}) - 1}{2} = \frac{x^{\frac{1}{2}}\theta(x) - 1}{2} = -\frac{1}{2} + \frac{x^{\frac{1}{2}}}{2} + x^{\frac{1}{2}}\omega(x).$$

Therefore,

$$\begin{aligned}\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right) \zeta(s) &= \int_1^\infty x^{s/2-1}\omega(x)\, dx + \int_1^\infty x^{-s/2-1}\left(-\frac{1}{2} + \frac{x^{\frac{1}{2}}}{2} + x^{\frac{1}{2}}\omega(x)\right) dx \\ &= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}}\right)\omega(x)\, dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}}\right)\omega(x)\, dx.\end{aligned}$$

Note that $\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} \leq \sum_{n=1}^{\infty} (e^{-\pi x})^n = O(e^{-\pi x})$ by geometric series. Therefore, the integral converges for all s.

One can show on any strip $S_{\delta,M}$ for $-\infty < \delta < M < \infty$, the integral uniformly converges, hence it is a holomorphic function. Solving for $\zeta(s)$, we have a pole at $s = 1$, and the pole at $s = 0$ on the RHS gets cancelled out by Γ. Therefore, the zeros on $\sigma < 0$ are $s = -2, -4, -6, \ldots$ by Equation B.1. Since this agrees with $\zeta(s)$ on $\sigma > 1$, we have constructed the unique analytic continuation of ζ to $\mathbb{C}$.

Moreover, since $s \mapsto 1 - s$ doesn't change the RHS above, we have the functional equation

$$\zeta(s) = \frac{\pi^{-\frac{1}{2}+s}\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)}\zeta(1-s).$$

Proof of Lemma B.5. By abuse of notation, we let $\theta(s) = \sum_{n\in\mathbb{Z}} e^{in^2\pi s}$. Therefore, we are considering $\theta(it) = \sum_{n\in\mathbb{Z}} e^{-n^2\pi t}$. Note that $f(x) = e^{-\pi x^2 t} \in \mathcal{S}(\mathbb{R})$ is a Schwarz function. By Poisson summation (A.9),

$$\sum_n f(n) = \sum_m \widehat{f}(m),$$

so

$$\theta(it) = \sum_m e^{-\pi m^2/t}/\sqrt{t} = \theta\left(\frac{i}{t}\right)/\sqrt{t},$$

which proves the result, letting $x = it$. ☐

One can also consider the **Riemann xi function**

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Lemma B.6

For $\sigma > 1$,

$$\xi(s) = \int_0^\infty \left(\frac{\theta(ix) - 1}{2}\right) x^{s/2}\,\frac{dx}{x}.$$

Proof. We have that

$$\int_0^\infty e^{-\pi n^2 x} x^{s/2}\,\frac{dx}{x} \overset{(u=\pi n^2 x)}{=} \int_0^\infty e^{-u}\left(\frac{u}{\pi n^2}\right)^{s/2}\frac{du}{u} = \pi^{-s/2} n^{-s}\Gamma\left(\frac{s}{2}\right).$$

Note that

$$\sum_{n=1}^{\infty}\int_0^\infty e^{-\pi n^2 x} x^{s/2}\,\frac{dx}{x}$$

is absolutely convergent, so we may apply Fubini's to get

$$\begin{aligned}\int_0^\infty \left(\frac{\theta(ix)-1}{2}\right) x^{s/2}\,\frac{dx}{x} &= \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 x} x^{s/2}\,\frac{dx}{x} \\ &= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 x} x^{s/2}\,\frac{dx}{x} \\ &= \sum_{n=1}^\infty \pi^{-s/2} n^{-s} \Gamma\left(\frac{s}{2}\right) \\ &= \pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) \\ &= \xi(s),\end{aligned}$$

as desired. □

References

[Apo98] T.M. Apostol. *Introduction to Analytic Number Theory*. Undergraduate Texts in Mathematics. Springer New York, 1998.

[Cha12] K. Chandrasekharan. *Introduction to Analytic Number Theory*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012.

[Fol99] G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 1999.