Swan's theorem

Pramana

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1. Introduction

Swan's theorem establishes a module structure on the set of smooth sections over a vector bundle $\pi\colon E \to M$.

This allows us to study algebra by means of vector bundles and vice-versa. We conclude with an introductory result in K-theory that follows from this theorem.

2. Projective modules

Recall that given a ring R, an **R-module** is an abelian group M together with an R-action $R \curvearrowright M$. Effectively, an R-module is the ring theory equivalent of a k-vector space for a field k. In fact, a k-module is a k-vector space.

A set $S = \{e_\alpha\}_\alpha \subseteq M$ is a **generating set** if the smallest submodule containing S is M. If M has a finite generating set, it is **finitely generated**.

An R-module M is a **free** if it has a linearly independent spanning set $\{e_\alpha\}_\alpha \in M$.

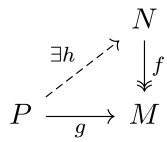
Example 2.1 –

$$\mathbb{Z}^n := \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{Z}\}$$

is a free $\mathbb{Z}$-module.

Definition 2.1. Let P be an R-module. P is a **projective module** is any of the following equivalent definitions hold:

1. There exists an R-module Q such that $P \oplus Q$ is free,
2. For any surjective R-module homomorphism $f\colon N \twoheadrightarrow M$ and R-module homomorphism $g\colon P \to M$, there exists an R-module homomorphism $h\colon P \to N$ such that $f \circ h = g$, i.e. the diagram



commutes.

Informally, this makes projective modules the "next best thing" to free modules.

Example 2.2 – If P is a free R-module, then since $P \oplus \{0\}$ is free, by definition (1), it is projective. Therefore, all free modules are projective.

3. Swan's theorem

First note that the set of (global) C^∞ sections $\Gamma(E)$ over a C^∞ vector bundle $\pi\colon E \to M$ forms a $C^\infty(M)$-module, since $s+t$ and fs are C^∞ sections for $s, t \in \Gamma(E)$ and $f \in C^\infty(M)$ [Tu10, Proposition 12.9].

Theorem 3.1 (Swan's theorem)**.** *If $\pi\colon E \to M$ is a smooth vector bundle, then $\Gamma(E)$ is a finitely-generated and projective $C^\infty(M)$-module. If M is connected, the converse holds.*

Swan's theorem also has a clean statement in the language of category theory.

Proposition 3.2 (Categorical Swan's theorem)**.** *Given a connected, smooth manifold M, the smooth section functor Γ from the category of smooth vector bundles over M to* *the category of finitely generated, projective $C^\infty(M)$-modules*

$$\Gamma\colon \mathsf{Vect}(M) \to C^\infty(M)\ \mathsf{Mod}_{\mathsf{pfg}}$$

is ***full, faithful,*** *and* ***essentially surjective.*** *In other words, the two categories are* ***equivalent.***

4. Applications

4.1. A projective but not free module

Example 2.2 illustrates that all free modules are projective, but we have not shown the converse. This example is from these notes with more details in the proof.

We start by noting that by Swan's theorem, $\Gamma(T\mathbb{S}^2)$, where $T\mathbb{S}^2$ is the tangent bundle of the 2-sphere, is projective.

Given two vector bundles $\pi\colon E \to M$, $\pi'\colon E' \to M$, we can form a direct sum of these vector bundles $E \oplus E'$, called a **Whitney sum**, by taking the pointwise direct sum of its fibers $E_p \oplus E'_p$ (recall each fiber is a vector space).

Fact 4.1. For vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M$, $\Gamma(E \oplus E') \cong \Gamma(E) \oplus \Gamma(E')$.

Fact 4.2. A vector bundle $\pi\colon E \to M$ is the trivial bundle (i.e. isomorphic to $M \times \mathbb{R}^{\dim M}$) if and only if $\Gamma(E)$ is free.

Note that $T\mathbb{S}^2$ is not a trivial bundle (a consequence of the hairy ball theorem), hence $\Gamma(T\mathbb{S}^2)$ is projective but not free.

We can take this further. A module P is **stably free** if there is a *free* module F such that $P \oplus F$ is free. By the first definition of projective module, any stably free module is projective.

Proposition 4.3. $T\mathbb{S}^2 \oplus \mathbb{R} \cong \mathbb{S}^2 \oplus \mathbb{R}^3$ *as vector bundles.*

Proof from [eEhee]. Viewing $\mathbb{S}^2$ as a surface in $\mathbb{R}^3$, we can write $T\mathbb{S}^2$ as all orthogonal vectors of $\mathbb{S}^2$ [Tu10, Problem 11.1]

$$T\mathbb{S}^2 = \left\{ (\mathbf{x}, \mathbf{v}) \mid \langle \mathbf{x}, \mathbf{v} \rangle = 0, \mathbf{x} \in \mathbb{S}^2, \mathbf{v} \in \mathbb{R}^3 \right\}.$$

We can write any vector $\mathbf{w} \in \mathbb{R}^3$ as the sum of a sum of vectors tangent and normal to the sphere, so there is a map

$$\begin{aligned} \mathbb{S}^2 \oplus \mathbb{R}^3 &\to T\mathbb{S}^2 \oplus \mathbb{R} \\ (\mathbf{x}, \mathbf{v}) &\mapsto (\mathbf{x}, \mathbf{v} - \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{x}, \langle \mathbf{x}, \mathbf{v} \rangle). \end{aligned}$$

This map is bijective and smooth, hence these vector bundles are isomorphic. □

Now we note that

$$\Gamma(T\mathbb{S}^2) \oplus \Gamma(\mathbb{R}) \cong \underbrace{\Gamma(\mathbb{S}^2) \oplus \Gamma(\mathbb{R}^3)}_{\text{free}}.$$

So $\Gamma(T\mathbb{S}^2)$ is a stably free module that is not free.

4.2. The start of K-theory

K-theory is the study of groups formed over other algebraic structures. It has found applications in string theory and in finding topological invariants (such as the *Chern character*) [nLa23].

Two branches of K-theory are topological K-theory and algebraic K-theory. They both rely on the Grothendieck completion of an abelian monoid (i.e. an abelian group dropping the assumption that elements are invertible).

Definition 4.1. Given an abelian monoid $(M, +)$, we can form the **Grothendieck group/completion** of M as the "smallest" (as in, it satisfies some universal property) abelian group A that can be formed out of M.

We construct the Grothendieck completion by creating the product monoid $M^2 = M \times M$ with an equivalence

$$(a_1, a_2) \sim (b_1, b_2)$$

if there exists $c \in M$ such that

$$a_1 + a_2 + c = b_1 + b_2 + c.$$

We use the following notation for the equivalence classes:

$$\begin{aligned} [a] - [b] &:= [a, b], \\ [a] &:= [a, 0], \\ -[b] &:= [0, b]. \end{aligned}$$

Example 4.4 – $\mathbb{N}_0 = \{0, 1, \ldots\}$ is an abelian monoid under addition. The Grothendieck completion of $\mathbb{N}_0$ is (up to isomorphism) $\mathbb{Z}$.

The set of all vector bundles over a manifold M become an abelian monoid under Whitney sums $\oplus$.

The Grothendieck completion of this abelian monoid is called the **K-theory of M**, denoted $K^0(M)$.

Motivated by Swan's theorem, we can also define the **K-theory of a ring R**. Consider the set of finitely generated projective R-modules over some ring R. They form an abelian monoid under direct sums. We then let the K-theory of R, $K_0(R)$, be the Grothendieck completion of this abelian monoid.

This means that topological K-theory is just a subset of algebraic K-theory (only looking at the case where $R = C^\infty(M)$).

The field of K-theory also analyzes the higher K^n, K_n groups, which are beyond the scope of this paper.

References

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