Modular Forms and Some Nice Applications

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Math Club

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I. $\mathbb{H}$ and $SL_2(\mathbb{Z})$

The Upper-Half Plane

We define the **upper-half plane** of the complex numbers as the subset

$$\mathbb{H} := \{x + iy : y > 0\} \subseteq \mathbb{C}.$$

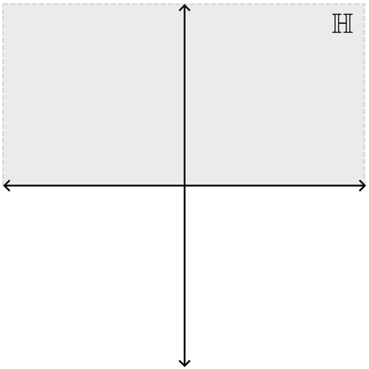

Figure: The upper half plane in $\mathbb{C}$.

Working With Matrices

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This formula is a little random, but it's the basis for the theory of modular forms.

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Example

$\mathrm{GL}_2(\mathbb{R})$ forms a group under matrix multiplication. For example, in linear algebra we learn that the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies

$$AI = IA = A.$$

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Example (Important)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d}.$$

Is a group action $\mathrm{GL}_2(\mathbb{R}) \curvearrowright \mathbb{H}$!

$\mathrm{SL}_2(\mathbb{Z})$

Define the **special linear group** as

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z},\ \det = 1 \right\}.$$

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If you are suspicious about inverses,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad - bc}_{=1}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

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This is a much smaller subset of $\mathrm{GL}_2(\mathbb{R})$, but we can still look at how it acts on $\mathbb{H}$ (with the formula before).

Generators of $SL_2(\mathbb{Z})$

Idea: Get a "basis" for $SL_2(\mathbb{Z})$.

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Theorem

The matrices

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It's harder for me to explain in words what these matrices represent.
Orbits of $z \in \mathbb{H}$.

$\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$

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What if we took the structure of $\mathbb{H}$, but simplified it by the group $\operatorname{SL}_2(\mathbb{Z})$?

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$$z \text{ is “the same” as } z' \text{ if there is a matrix taking } z \text{ to } z'.$$

What I've just described is **quotienting** $\mathbb{H}$ by $\operatorname{SL}_2(\mathbb{Z})$, which we will denote $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$.

The fundamental domain

We can now just look at one of these regions:

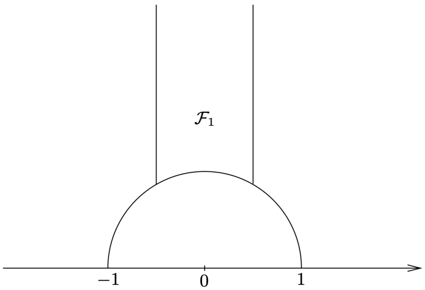

Figure: The upper-half plane, quotiented by $SL_2(\mathbb{Z})$. [Stein, Shakarchi *Complex Analysis*]

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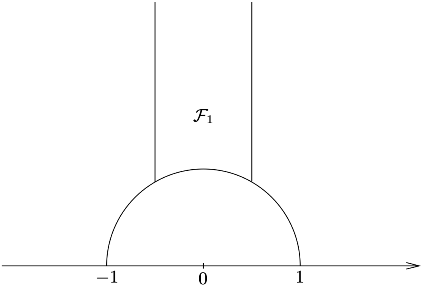

Figure: The upper-half plane, quotiented by $SL_2(\mathbb{Z})$. [Stein, Shakarchi *Complex Analysis*]

We call $\mathcal{F}$ the **fundamental domain**.

II. Modular Forms

What are modular forms?

A question we may ask is what “nice” complex functions satisfy

$$f\left(\frac{az+b}{cz+d}\right) = f(z)$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$? These are **modular functions**.

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Fourier Things

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Theorem

Let $q(z) := e^{2\pi iz}$. We can express any modular form as the Fourier series

$$f(q) = f(q(z)) = \sum_{n=0}^{\infty} a_n q(z)^n = \sum_{n=0}^{\infty} a_n q^n.$$

*This is called the q-**expansion** of f.*

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Question: What does $q = 0$ correspond to?

Vector Spaces

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Remark

*So $\mathcal{M} := \bigoplus_{k \geq 0} M_k$ has a **graded module** structure.*

Finite-dimensionality

Theorem

M_k is finite-dimensional for all k. In fact,

$$\dim M_k = \begin{cases} \left[\frac{k}{12}\right] + 1 & \text{if } k \text{ even, } k \not\equiv 2 \pmod{12}, \\ \left[\frac{k}{12}\right] & \text{if } k \text{ even, } k \equiv 2 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

Some quick examples of modular forms

Definition

For $k \geq 4$ even, define the **weight** k **Eisenstein series** as

$$G_k(z) := \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}.$$

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For the modularity condition, you just need to prove it for the matrices $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, since they generate $\mathrm{SL}_2(\mathbb{Z})$ (exercise).

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$$G_k(q) = 2\zeta(k) - q(\text{other terms...}).$$

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Upshot: Sums and products of these E_k's form a basis for any M_k!

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Looking at $q = 0$, $C = 1$.

III. Finally, some applications!

Translation surfaces

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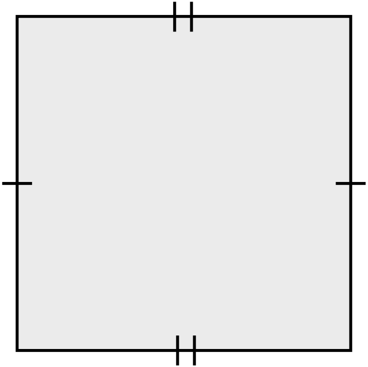

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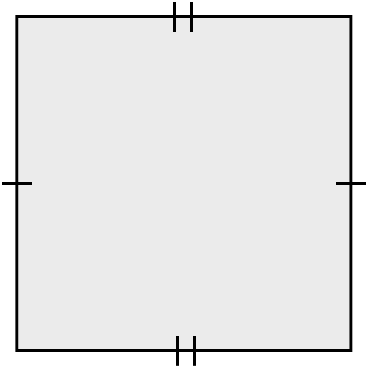

Figure: A translation surface. What surface is this?

Parallelogram torus $\leftrightarrow \mathbb{C}$

Similarly, we can create a torus for any parallelogram.

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Similarly, we can create a torus for any parallelogram. With scaling and rotation, we can associate any $z \in \mathbb{H}$ to a torus.

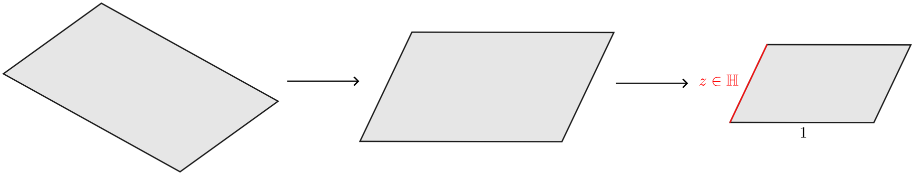

Figure: Associating a complex number $\mathbb{H}$ to every torus.

Spaces of translation surfaces

Let's see what happens when we apply $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ to a square torus.

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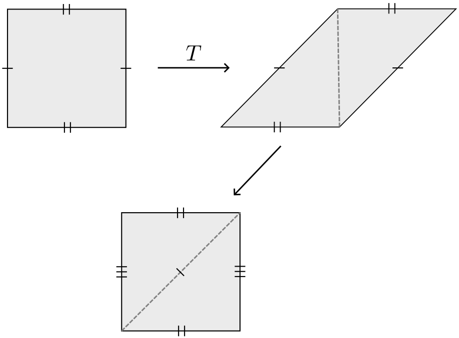

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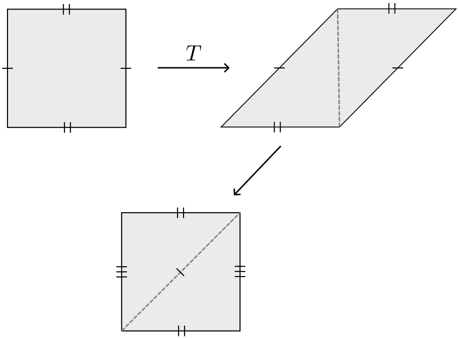

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Since $-\frac{1}{i} = i$, the torus doesn't change under S.

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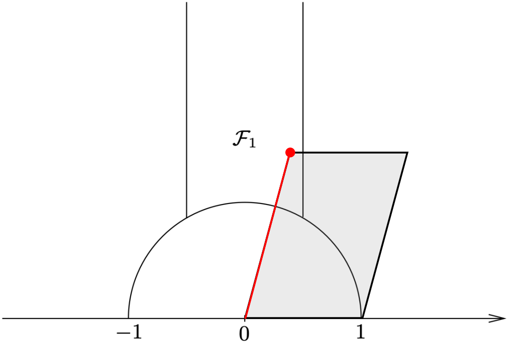

Figure: A translation surface associated to each point in $\mathcal{F}$.

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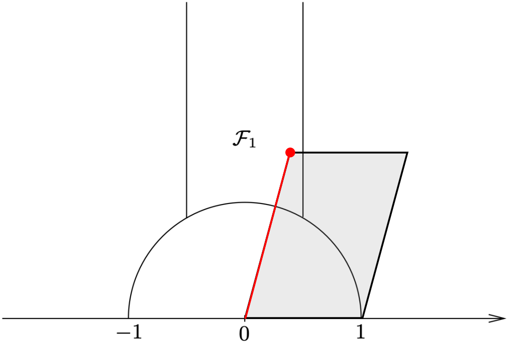

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This is called the **stratum** $\mathcal{H}(\varnothing)$.

Jacobi's four squares theorem

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Theorem (Jacobi, 1834)

In fact, the number of ways to write an integer n as the sum of four squares is

$$r_4(n) = \begin{cases} 8\sigma_1(n) & n \text{ odd}, \\ 24\sigma_1(n_{\mathrm{odd}}) & n \text{ even}, \end{cases}$$

where $\sigma_1(n)$ is the sum of the divisors of n and n_{odd} is the odd integer so that $n = 2^k \cdot n_{\mathrm{odd}}$.

We can prove Jacobi's theorem with modular forms!

Definition

Let $k \in \mathbb{Z}$. A **modular form of weight** k for $\mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that

1. f is holomorphic,
2. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,
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Let $k \in \mathbb{Z}$. A **modular form of weight** k for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that

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2. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$,
3. $f(z)$ is bounded as $\mathrm{Im}\, z \to \infty$.

The vector space of all modular forms of weight k for Γ is denoted $M_k(\Gamma)$.

We can prove Jacobi's theorem with modular forms!

- We consider the function

$$\theta(z) := \sum_{m \in \mathbb{Z}} e^{2\pi i m^2 \tau} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

The coefficient of q^k counts the number of ways to write $k = a^2$ for integers a.

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- It turns out

$$\theta(z)^4 = 1 + 8q + 24q^2 + \cdots$$

  is a modular form of weight 2 on $\Gamma_0(4) \subseteq \mathrm{SL}_2(\mathbb{Z})$. Moreover, the coefficient of q^k counts the number of ways to add four squares to k.

We can prove Jacobi's theorem with modular forms!

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$$\theta(z) := \sum_{m\in\mathbb{Z}} e^{2\pi i m^2\tau} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

  The coefficient of q^k counts the number of ways to write $k = a^2$ for integers a.

- It turns out

$$\theta(z)^4 = 1 + 8q + 24q^2 + \cdots$$

  is a modular form of weight 2 on $\Gamma_0(4) \subseteq \mathrm{SL}_2(\mathbb{Z})$. Moreover, the coefficient of q^k counts the number of ways to add four squares to k.

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- Problem: it's hard to expand this power series and get the values.
- A similar result to before shows that $M_2(\Gamma_0(4))$ is finite dimensional, so we instead use a basis to determine the coefficients.

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- So take basis polynomials for $M_2(\Gamma_0(4))$:

$$f_1(z) = 1 + 24\sum_{n=1}^{\infty} \sigma_1(n_{\mathrm{odd}})q^n = 1 + 24q + \cdots$$

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$$\theta(z)^4 = \frac{1}{3}f_1(z) + \frac{2}{3}f_2(z).$$

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- But f_1 and f_2 have a simple form, which gives us the formula.

Some properties of ζ

Definition

The **Riemann zeta function** is defined[a] as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

[a] For all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$. Also, defined for all $s \in \mathbb{C}$ by **analytic continuation**.

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Proposition (Basel Problem)

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

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What about other values of $\zeta(s)$?

Recall:

Proposition

$$G_k(q) := \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k} = 2\zeta(k) - q(\textit{other terms}\ldots).$$

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Proof (continued).

Differentiating k times, we get

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q.$$

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So,

$$G_k(z) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^k} + \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$$

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Finally,

$$G_k(z) = 2\zeta(k) + \frac{2\cdot(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1}q^{nm}$$

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Theorem

If $k > 0$ and $f \in M_k$ has q expansion $\sum_{n=0}^{\infty} a_n q^n$ with $a_n \in \mathbb{Q}$ for $n \geq 1$, then $a_0 \in \mathbb{Q}$.

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Proof.

Partially technical... (axiom of choice/field theory). Uses the fact that E_4 and E_6 form a basis for $\mathcal{M}$. □

Rationality of $\zeta(k)$

Now notice that

$$\frac{G_k(z)}{2(2\pi i)^k/(k-1)!} = \frac{\zeta(k)}{(2\pi i)^k/(k-1)!} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

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Corollary

$\zeta(k)$ is a rational multiple of π^k for $k \geq 8$ even.

$$\zeta(8) = \frac{\pi^8}{9450}$$

$$\zeta(10) = \frac{\pi^{10}}{93555}$$

$$\zeta(12) = \frac{691\pi^{12}}{638512875}$$

Thank you!