The Ramanujan Tau Function

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MA4263

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Notice a pattern???

In 1916, Ramanujan considered the following series:

$$\Delta := q \prod_{n \geq 1} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots =: \sum_{n=1}^{\infty} \tau(n) q^n.$$

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$$\underbrace{(-24)}_{\tau(2)}\underbrace{(252)}_{\tau(3)} = \underbrace{-6048}_{\tau(6)}$$
$$\underbrace{(-24)}_{\tau(2)}\underbrace{(4830)}_{\tau(5)} = \underbrace{-115920}_{\tau(12)}.$$

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Define a *Dirichlet L-function*, which is similar to the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$, but replace the numerator with some "nice" multiplicative function $a(n)$:

$$L(s, A) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

In our case, let

$$L(s, \Delta) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.$$

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Also, remember ζ has an *Euler product*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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2. *$L(s, \Delta)$ has an Euler product:*

$$L(s, \Delta) = \prod_p \left(1 - \frac{\tau(p)}{p^s} + \frac{p^{11}}{p^{2s}}\right)^{-1}.$$

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Theorem (Mordell, 1917)

Yes.

Theorem (Hecke, 1937)

Yes, and there is a richer theory that extends this to even more "τ-like" functions.

The minimal modular forms theory needed

Let the **upper-half plane** of the complex numbers be the subset

$$\mathbb{H} := \{x + iy : y > 0\} \subseteq \mathbb{C}.$$

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Example

$$\begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \cdot z = \begin{bmatrix} 4 & \\ & 2 \end{bmatrix} \cdot z = \frac{4z + 0}{0z + 2} = 2z.$$

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Orbits of $z \in \mathbb{H}$.

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Consider functions $f : \mathbb{H} \to \mathbb{C}$ that are "almost" $\mathrm{SL}_2(\mathbb{Z})$ invariant:

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$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

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Important consequence: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, so

$$f(z+1) = f\left(\frac{1z+1}{0z+1}\right) = (1)^k f(z) = f(z).$$

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So we can do Fourier analysis!

Important result: The space of modular forms of weight k, M_k, is finite-dimensional for all k.

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Δ appears

It turns out

$$\Delta(z) := e^{2\pi iz} \prod_{n\geq 1} \left(1 - e^{2\pi inz}\right)^{24}$$

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$$\Delta(q) := q \prod_{n\geq 1} (1 - q^n)^{24} \qquad (q(z) = e^{2\pi i z})$$

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Why Hecke operators?

Issue: modular forms are only "almost invariant" under $\gamma \in \mathsf{SL}_2(\mathbb{Z})$.
What about matrices in $\mathsf{GL}_2(\mathbb{Q})^+$?

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What about matrices in $\mathrm{GL}_2(\mathbb{Q})^+$?
First, rephrase the modularity condition. Consider the action of $g \in \mathrm{GL}_2(\mathbb{R})$ on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$(f|g)(z) \coloneqq (\det g)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

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$$(f|g)(z) := (\det g)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Then the modularity condition is $f|\gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Do averaging

One attempt: average.

$$T_n f := \sum_{\substack{g \in \mathrm{GL}_2(\mathbb{Q})^+ \\ \det g = n}} f|g.$$

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$$T_n f := \sum_{\substack{g \in \mathrm{GL}_2(\mathbb{Q})^+ \\ \det g = n}} f|g.$$

Issue: infinitely many matrices! Ok, but $f|\gamma = f$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, so instead take the sum

$$T_n f := \sum_{\substack{g \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{GL}_2(\mathbb{Q})^+ \\ \det g = n}} f|g.$$

An example

So what does a set of representatives for $SL_2(\mathbb{Z}) \setminus \{\gamma \in GL_2(\mathbb{Q})^+ : \det\gamma = n\}$ look like?

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Example ($n = 2$)

Any matrix with determinant 2 represents the same $\mathrm{SL}_2(\mathbb{Z})$ left coset as an upper triangular matrix of the form $\begin{bmatrix} 2 & * \\ & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & * \\ & 2 \end{bmatrix}$.

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$$\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ & 2 \end{bmatrix},$$

yields a complete set of representatives.

$$T_n f := \sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{GL}_2(\mathbb{Q})^+ \\ \det \gamma = n}} f|\gamma.$$

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Theorem

The operator T_n is a linear operator on M_k and S_k for all $n \geq 2$. Moreover, for all k, there exists an orthonormal basis for M_k and S_k with respect to the operators $\{T_n : n \geq 2\}$.

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We call elements of this basis *Hecke eigenforms*.
It will be convenient to normalize T_n so that

$$T_n f = n^{1-k/2} \lambda_f(n) f$$

for a fixed eigenform f.

Proving Ramanujan's τ conjecture

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$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} \in S_{12},$$

which is a 1-dimensional vector space.

Proving Ramanujan's τ conjecture

We explicitly apply Hecke operators to Δ as an example. We have that

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} \in S_{12},$$

which is a 1-dimensional vector space. Therefore, Δ is a Hecke eigenform.

We have that $T_2\Delta = \Delta\Big| \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} + \Delta\Big| \begin{bmatrix} 1 & 0 \\ & 2 \end{bmatrix} + \Delta\Big| \begin{bmatrix} 1 & 1 \\ & 2 \end{bmatrix}.$

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$$\Delta\Big| \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}(z) = (2)^{12/2}\Delta(2z) = 2^6\Delta(q^2)$$

$$\Delta\Big| \begin{bmatrix} 1 & 0 \\ & 2 \end{bmatrix}(z) = (2)^{12/2}(2)^{-12}\Delta\left(\frac{z}{2}\right) = 2^{-6}\Delta(\sqrt{q})$$

$$\Delta\Big| \begin{bmatrix} 1 & 1 \\ & 2 \end{bmatrix}(z) = (2)^{12/2}(2)^{-12}\Delta\left(\frac{z+1}{2}\right) = 2^{-6}\Delta(-\sqrt{q}).$$

Summing,

$$
\begin{array}{lcccccccccc}
\Delta\Big|\begin{bmatrix}2 & \\ & 1\end{bmatrix}(q) & = & 0q^{1/2} & + & 0q^{1} & + & 0q^{3/2} & + & 2^{6}\tau(1)q^{2} & + & \cdot \\
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\Delta\Big|\begin{bmatrix}1 & 1\\ & 2\end{bmatrix}(q) & = & -2^{-6}\tau(1)q^{1/2} & + & 2^{-6}\tau(2)q^{1} & + & -2^{-6}\tau(3)q^{3/2} & + & 2^{-6}\tau(4)q^{2} & + & \cdot \\
\hline
T_2\Delta & = & 0q^{1/2} & + & 2^{-5}\tau(2)q^{1} & + & 0q^{3/2} & + & (2^{6}\tau(1)+2^{-5}\tau(4))q^{2} & + & \cdot
\end{array}
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\end{array}
$$

So

$$
T_2\Delta[q^n] = \begin{cases} 2^{6}\tau(n/2) + 2^{-5}\tau(2n) & \text{if } n \text{ is even,} \\ 2^{-5}\tau(2n) & \text{if } n \text{ is odd.} \end{cases}
$$

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by the formula before. Hence,

$$\lambda(2) = -24 = \tau(2).$$

We can check now that $\tau(2)\tau(3) = \tau(6)$. Indeed,

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This proves even more. If m is odd (coprime with 2), then

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In general, the process is showing that $\lambda(n) = \tau(n)$, and then $\lambda(n)\tau(m) = \tau(nm)$ for all m coprime to n.

Thank you!

Further reading

For more information, see Bump's *Automorphic Forms and Representations*.

I have written accompanying notes (with more details and proofs), but they are not done yet.