Ramanujan's Tau Function

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In 1916, Ramanujan considered the following series, whose nth power series coefficient (for $n \geq 1$), he denoted $\tau(n)$:

$$q\prod_{n\geq 1}(1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots =: \sum_{n=1}^{\infty} \tau(n)q^n.$$

The goal of this note is to prove that τ is multiplicative: if $(m,n) = 1$, then $\tau(nm) = \tau(n)\tau(m)$, and that the L-function attached to τ, defined as

$$L(s,\Delta) := \sum_{n=1}^{\infty} \tau(n)n^{-s},$$

has the following Euler product factorization:

$$L(s,\Delta) = \prod_p \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1}.$$

1. Basic modular forms

Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})^+$ (that is, 2×2 matrices in $\mathbb{R}$ with positive determinant). Suppose f is a function from $\mathbb{H} := \{z : \mathrm{Re}(z) > 0\}$ to $\mathbb{C}$. Let $k \in \mathbb{Z}$. We define a right action of $GL_2(\mathbb{R})^+$ on the space of functions from $\mathbb{H} \to \mathbb{C}$ as follows:

$$(f|\gamma)(z) := (\det\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Definition 1.1

We say a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is a **modular form of weight k** if $f|\gamma = f$ for all $\gamma \in SL_2(\mathbb{Z})$ (the so-called *modularity condition*), and if f is *holomorphic at the cusp at ∞*.

Holomorphic at the cusp at ∞ just means that as $z \to i\infty$, $f(z)$ is bounded.

We note that $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \in SL_2(\mathbb{Z})$, so the modularity condition implies that

$$f(z+1) = f(z).$$

With this, f has a Fourier transform, and we may write it as

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}.$$

Defining $q(z) = e^{2\pi i z}$ (which by abuse of notation we will just write as q), we have $f(z) = \sum_{n \geq 0} a_n q^n$. This is the **q-expansion** of f, and is where connections to number theory arise.

1.1. Congruence subgroups

Definition 1.2

If N is a natural number (other than 1), let

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}.$$

If $N = 1$, then let $\Gamma(1) := SL_2(\mathbb{Z})$.

Definition 1.3

If $\Gamma \leq SL_2(\mathbb{Z})$ is a subgroup that contains $\Gamma(N)$ for some N, then call Γ a **congruence subgroup**.

Since $\Gamma(N)$ has finite index in $\Gamma(1)$, this automatically means Γ has finite index in $\Gamma(1)$.

We need the following result from modular forms:

Theorem 1.1 (The space of modular forms of weight k is finite-dimensional)

Let $M_k(\Gamma(1))$ be the space of all modular forms of weight k. If k is even, and we write it as $12j + r$ for $j \geq 0$ and $0 \leq r \leq 10$, then

$$\dim M_{12j+r}(\Gamma(1)) = \begin{cases} j+1 & \text{if } r \in \{0, 4, 6, 8, 10\}, \\ j & \text{if } r = 2. \end{cases}$$

We may also want to consider **cusp forms**, which are modular forms whose q-expansion has $a_0 = 0$. We let the space of weight k modular cusp forms be $S_k(\Gamma(1))$. One can show that

Proposition 1.2

$\Delta(q)$ is a cusp form of weight 12 and generates $S_{12}(\Gamma(1))$. In particular, $S_{12}(\Gamma(1))$ is one-dimensional.

2. Hecke operators

This section closely follows [Bum97, pp. 41-54].

Lemma 2.1
The conjugate of a congruence subgroup Γ by $\alpha \in GL_2(\mathbb{Q})^+$ contains some $\Gamma(M)$ for $M > 0$.

Proof. Let $\Gamma(N) \subseteq \Gamma$. There are integers M_1 and M_2 such that $M_1\alpha, M_2\alpha^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$. Let $M = NM_1M_2$. For $\gamma \in \Gamma(M)$, we may write it as $I + Mg$ for some $g \in \mathrm{Mat}_2(\mathbb{Z})$. But then

$$\alpha\gamma\alpha^{-1} = I + N(M_1\alpha)g(M_2\alpha^{-1}) \in \Gamma(N) \subseteq \Gamma.$$

Therefore,

$$\gamma = \alpha^{-1}(\alpha\gamma\alpha^{-1})\alpha \in \alpha^{-1}\Gamma\alpha. \qquad \square$$

As a quick corollary, $\alpha^{-1}\Gamma\alpha \cap \Gamma(1)$ has finite index in $\Gamma(1)$, since it is a congruence subgroup.

Proposition 2.2
If $\alpha \in GL_2(\mathbb{Q})^+$ is a matrix, then the double coset $\Gamma(1)\alpha\Gamma(1)$ is the finite union of disjoint right cosets:

$$\Gamma(1)\alpha\Gamma(1) = \bigcup_{i=1}^{N} \Gamma(1)\alpha_i,$$

where $N = [\Gamma(1) : \alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1)]$.

Proof. Note that N is finite because $\alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1)$ is a congruence subgroup.[1] Right translation by α^{-1} yields a bijection

$$\Gamma(1) \setminus \Gamma(1)\alpha\Gamma(1) \longleftrightarrow \Gamma(1) \setminus \Gamma(1)\alpha\Gamma(1)\alpha^{-1} \cong (\alpha\Gamma(1)\alpha^{-1} \cap \Gamma(1)) \setminus \alpha\Gamma(1)\alpha^{-1}.$$

Conjugating by α gives the result. $\square$

[1] WLOG, we assume α has integer entries. Then $\alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1)$ contains $\Gamma(m)$ for $m = \det\alpha$. [To prove that $\Gamma(m) \subseteq \alpha^{-1}\Gamma(1)\alpha$, which is equivalent to $\alpha^{-1}\Gamma(m)\alpha \subseteq \Gamma(1)$, write $\gamma \in \Gamma(m)$ as $I + mA$ for some integer-valued matrix A.]

For $\alpha \in GL_2(\mathbb{Q})^+$ we define the **Hecke operator** T_α as

$$f|T_\alpha \coloneqq \sum_i f|\alpha_i.$$

[This defines a linear operator from $M_k(\Gamma(1))$ to itself, with invariant subspace $S_k(\Gamma(1))$.]

If $\alpha, \beta \in GL_2(\mathbb{Q})^+$, then

$$f|T_\alpha|T_\beta = \sum_{i,j} f|\alpha_i\beta_j = \sum_{\sigma\in\Gamma(1)\setminus GL_2(\mathbb{Q})^+} m(\alpha, \beta; \sigma)f|\sigma,$$

where $m(\alpha, \beta; \sigma)$ just counts the number of pairs (i, j) such that $\sigma \in \Gamma(1)\alpha_i\beta_j$ in the first sum above. But, in fact, $m(\alpha, \beta; \sigma)$ only depends on a representative of $\Gamma(1)\sigma\Gamma(1)$. [We prove this as follows: if $\gamma \in \Gamma(1)$, then

$$m(\alpha, \beta; \sigma\gamma) = \{(i, j) : \sigma\gamma \in \Gamma(1)\alpha_i\beta_j\} = \#\left\{(i, j) : \sigma \in \Gamma(1)\alpha_i\beta_j\gamma^{-1}\right\}.$$

Since $\Gamma(1)\beta\Gamma(1) = \bigsqcup_j \Gamma(1)\beta_j$, right-multiplication by $\gamma^{-1} \in \Gamma(1)$ permutes these right cosets. Thus, there exists a permutation π of the indices j and elements $\gamma_j \in \Gamma(1)$ such that

$$\beta_j\gamma^{-1} = \gamma_j\beta_{\pi(j)}.$$

Similarly, for each fixed j, right-multiplication by $\gamma_j \in \Gamma(1)$ permutes the cosets $\{\Gamma(1)\alpha_i\}_i$. Hence, there exists a permutation ρ_j of the indices i and elements $\gamma'_{i,j} \in \Gamma(1)$ such that

$$\alpha_i\gamma_j = \gamma'_{i,j}\alpha_{\rho_j(i)}.$$

Substituting these identities gives

$$\alpha_i\beta_j\gamma^{-1} = \alpha_i\gamma_j\beta_{\pi(j)} = \gamma'_{i,j}\alpha_{\rho_j(i)}\beta_{\pi(j)} \in \Gamma(1)\alpha_{\rho_j(i)}\beta_{\pi(j)}.$$

Since the map $(i, j) \mapsto (\rho_j(i), \pi(j))$ is a bijection on the finite set of index pairs, we conclude that

$$\#\left\{(i, j) : \alpha_i\beta_j\gamma^{-1} \in \Gamma(1)\sigma\right\} = \#\left\{(i, j) : \alpha_{\rho_j(i)}\beta_{\pi(j)} \in \Gamma(1)\sigma\right\} = m(\alpha, \beta; \sigma).$$

]

Therefore, we may rewrite this to

$$f|T_\alpha|T_\beta = \sum_{i,j} f|\alpha_i\beta_j = \sum_{\sigma\in\Gamma(1)\backslash \mathrm{GL}_2(\mathbb{Q})^+/\Gamma(1)} m(\alpha, \beta; \sigma)f|T_\sigma.$$

Definition 2.1

The **Hecke algebra** for $\mathrm{SL}_2(\mathbb{Z})$, denoted $\mathcal{R}$, is the free abelian group generated by the symbols T_α, for each representative α of each double coset in $\Gamma(1) \backslash \mathrm{GL}_2(\mathbb{Q})^+/\Gamma(1)$.

We define multiplication in $\mathcal{R}$ by

$$T_\alpha \cdot T_\beta := \sum_{\sigma\in\Gamma(1)\backslash \mathrm{GL}_2(\mathbb{Q})^+/\Gamma(1)} m(\alpha, \beta; \sigma)T_\sigma,$$

with $m(\alpha, \beta; \sigma)$ as defined above.

Note that $\mathcal{R}$ obviously acts on the space of modular (or cusp) forms of weight k by $f \mapsto f|T_\alpha$. We now give an explicit set of coset representatives that we can take this sum over.

Proposition 2.3
A complete set of coset representatives for

$$\Gamma(1) \backslash \mathrm{GL}_2(\mathbb{Q})^+ / \Gamma(1)$$

consists of the diagonal matrices $\mathrm{diag}(d_1, d_2)$, where $d_1, d_2 \in \mathbb{Q}^+$ and d_1/d_2 is an integer.

Proof. Let $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, and let N be the smallest positive integer such that $N\alpha$ has integer entries. By the classification of modules over a PID, there is a basis ξ_1, ξ_2 of $\mathbb{Z}^2$ such that the columns of $N\alpha$ are $D_1\xi_1$ and $D_2\xi_2$, where $D_2 \mid D_1$ and $D_1, D_2 > 0$. Let ξ be the matrix with columns ξ_1 and ξ_2. Then $\det(\xi) \in \mathbb{Z}^\times = \{\pm 1\}$. We may assume $\det(\xi) = 1$ WLOG by replacing ξ_1 with $-\xi_1$. Hence, $\xi \in \mathrm{SL}_2(\mathbb{Z})$.

Note that $\mathrm{diag}(D_1, D_2)\xi$ spans the same submodule of $\mathbb{Z}^2$ as $N\alpha$. Therefore, there exists a matrix $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$\gamma N\alpha = \mathrm{diag}(D_1, D_2)\xi.$$

We have $\det(\gamma) = 1$ by noting the determinant of all other matrices in this equation are positive. Therefore, taking $d_i = D_i/N$,

$$\alpha = \gamma^{-1}\mathrm{diag}(d_1, d_2)\xi,$$

which implies it is the same double coset as $\mathrm{diag}(d_1, d_2)$.

We now show no two such coset representatives are the same if the matrices $\mathrm{diag}(d_1, d_2)$ are distinct. By the uniqueness of invariant factors, it suffices to check that if we chose some other N' such that $\alpha N' \in \mathrm{Mat}_2(\mathbb{Z})$, then d_1 and d_2 are the same. [Suppose the contrary. Note that $N \mid N'$ because N is minimal. Repeating the same proof as above, we note that the invariant factors D_1' and D_2' of $N'\alpha$ are unique up to multiplication by -1. But in particular, $D_i' = D_i N'/N$ work as invariant factors (because $N'\alpha = (N'/N)(N\alpha)$ spans the same submodule as $(N'/N)\mathrm{diag}(D_1, D_2)\xi$), so $D_1'/N' = d_1$ and $D_2'/N' = d_2$ in the proof before.] □

Corollary 2.4
For $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, $\Gamma(1)\alpha\Gamma(1) = \Gamma(1)\alpha^\top\Gamma(1)$.

Lemma 2.5
Given some $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, there exist some α_i such that

$$\Gamma(1)\alpha\Gamma(1) = \bigcup_{i=1}^{N} \Gamma(1)\alpha_i = \bigcup_{i=1}^{N} \alpha_i\Gamma(1),$$

where both unions are through distinct cosets.

Proof. Note that

$$\Gamma(1)\alpha\Gamma(1) \overset{(2.4)}{=} \Gamma(1)\alpha^\top\Gamma(1) = (\Gamma(1)\alpha\Gamma(1))^\top \overset{(2.2)}{=} \left(\bigcup_{i=1}^N \Gamma(1)\alpha_i\right)^\top = \bigcup_{i=1}^N \alpha_i^\top\Gamma(1).$$

Since α_i and $\alpha_i^\top$ generate the same double coset,

$$\Gamma(1)\alpha_i \cap \alpha_i^\top\Gamma(1) \neq \varnothing,$$

so choose some β_i in this intersection for all i. □

Theorem 2.6

The Hecke algebra $\mathcal{R}$ is commutative.

Proof. Let $\alpha, \beta \in GL_2(\mathbb{Q})^+$. For $\sigma \in GL_2(\mathbb{Q})^+$, let

$$\deg(\sigma) \coloneqq |\Gamma(1) \setminus \Gamma(1)\sigma\Gamma(1)| = |\Gamma(1)\sigma\Gamma(1)/\Gamma(1)|\,.$$

In other words, the number of α_i in Lemma 2.5. Write $\Gamma(1)\sigma\Gamma(1) = \bigcup_k \Gamma(1)\sigma_k$. Then $\sigma \in \Gamma(1)\alpha_i\beta_j\Gamma(1)$ if and only if some σ_k is in $\Gamma(1)\alpha_i\beta_j$. Therefore, the number of (i, j) such that $\sigma \in \Gamma(1)\alpha_i\beta_j\Gamma(1)$ is

$$\sum_k m(\alpha, \beta; \sigma_k) = \sum_k m(\alpha, \beta; \sigma) = \deg(\sigma) m(\alpha, \beta; \sigma).$$

Let α_i and β_i be representatives of the double cosets of α and β, respectively, so that Lemma 2.5 holds. We then have

$$\begin{aligned} \Gamma(1)\alpha\Gamma(1) &= \bigcup_i \Gamma(1)\alpha_i = \bigcup_i \Gamma(1)\alpha_i^\top, \\ \Gamma(1)\beta\Gamma(1) &= \bigcup_j \Gamma(1)\beta_j = \bigcup_j \Gamma(1)\beta_j^\top, \end{aligned}$$

so

$$\begin{aligned} m(\alpha, \beta; \sigma) &= \frac{1}{\deg(\sigma)} \# \left\{(i, j) : \sigma \in \Gamma(1)\alpha_i^\top\beta_j^\top\Gamma(1)\right\} \\ &= \frac{1}{\deg(\sigma)} \# \left\{(i, j) : \sigma^\top \in \Gamma(1)\beta_j\alpha_i\Gamma(1)\right\} \\ &= m(\beta, \alpha; \sigma^\top). \end{aligned}$$

Therefore, $T_\sigma = T_{\sigma^\top}$, which implies that the Hecke algebra is commutative. □

Therefore, we write $T_\alpha f$ instead of $f|T_\alpha$, since there is no distinction between the left and right action.

We say that f is a **congruence form** (resp. **congruence cusp form**) if it is a modular (resp. cusp) form for *some* congruence subgroup.

Suppose f and g are congruence cusp forms, which are both cusp forms for $\Gamma(N)$. We define the **Petersson inner product** on the space of congruence cusp forms of weight k as

$$\langle f, g \rangle := \frac{1}{[\Gamma(1) : \Gamma(N)]} \int_{\Gamma(N)\backslash\mathbb{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

Theorem 2.7

The operators T_α on $S_k(\Gamma(1))$ are self-adjoint with respect to the Petersson inner product.

Proof. Since $z \mapsto \alpha^{-1}(z)$ is a measure-preserving isometry of $\mathbb{H}$ with respect to the measure $\frac{dxdy}{y^2}$, this change of variables induces the equality

$$\langle f|\alpha, g \rangle = \left\langle f, g|\alpha^{-1} \right\rangle.$$

The LHS implies that the inner product is invariant under $\alpha \mapsto \gamma\alpha$, where $\gamma \in \Gamma(1)$, and similarly, the RHS implies the inner product is invariant under $\alpha \mapsto \alpha\gamma$. Therefore, this inner product only depends on the coset $\Gamma(1)\alpha\Gamma(1)$.

Let $T_\alpha f = \sum_i f|\alpha_i$. Using the fact that $\alpha_i \in \Gamma(1)\alpha\Gamma(1)$ for all i, we have

$$\begin{aligned} \langle T_\alpha f, g \rangle &= \sum_i \langle f|\alpha_i, g \rangle \\ &= \deg(\alpha) \langle f|\alpha, g \rangle \\ &= \deg(\alpha) \left\langle f, g|\alpha^{-1} \right\rangle \\ &= \deg(\alpha) \left\langle f, g|(\alpha^{-1})^\top \right\rangle. \end{aligned}$$

Let $\beta = \det(\alpha)(\alpha^{-1})^\top$. Since scalar matrices act trivially on modular forms, we have

$$\langle T_\alpha f, g \rangle = \deg(\alpha) \langle f, g|\beta \rangle.$$

Since $\beta = S\alpha S^{-1}$, where $S = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$, $\Gamma(1)\beta\Gamma(1) = \Gamma(1)\alpha\Gamma(1)$.

Since β generates the same double coset as α, the Hecke operator associated with β is identically T_α. Reversing our earlier logic to sum over the right coset representatives β_j of $\Gamma(1)\beta\Gamma(1)$, we obtain

$$\deg(\alpha) \langle f, g|\beta \rangle = \sum_j \langle f, g|\beta_j \rangle = \langle f, T_\beta g \rangle.$$

Since $T_\beta = T_\alpha$, we conclude that

$$\langle T_\alpha f, g \rangle = \langle f, T_\alpha g \rangle,$$

proving that T_α is self-adjoint. □

Lemma 2.8

If S is a set of commuting, normal operators on a finite dimensional space V, then there exists a basis for V which is a simultaneous eigenbasis for all $T \in S$.

Proof (see [Axl97, Exr. 7B.9,12,16]). We first prove that if $T \in \mathcal{L}(V)$ is normal and U is a T-invariant and T^*-invariant subspace of V. Then $T|_U$ is a normal operator. This is true because $\|Tu\| = \|T|_U u\| = \|(T^*)|_U u\| = \|T^* u\|$ for all $u \in U$, since this holds for all $v \in V$.

We induct on $\dim V = n$. This obviously holds for $n = 1$. For $n > 1$, if all operators are a multiple of the identity, then any orthonormal basis will suffice. Otherwise, fix some $T \in \mathcal{E}$ that has a non-trivial proper eigenspace $E := E(\lambda, T)$. For each other $S \in \mathcal{E} - \{T\}$, E is S-invariant, since S and T commute. But then it further follows that E is also S^*-invariant because S^* and T commute by Fuglede's theorem.[1] Therefore, $S|_E$ is a normal operator. By the inductive hypothesis,

$$\{S|_E : S \in \mathcal{E}\}$$

have an orthonormal basis such that all matrices are diagonal. We now consider $E^\perp$. If $u \in E^\perp$, then for $v \in E$,

$$\langle Su, v\rangle = \langle u, S^* v\rangle = 0,$$

where the last line follows because $S^* v \in E$. Similarly,

$$\langle v, S^* u\rangle = \langle Sv, u\rangle = 0,$$

because $Sv \in E$. Therefore, $E^\perp$ is also S-invariant and S^*-invariant for all $S \in \mathcal{E}$. Hence, the inductive hypothesis also applies to the set

$$\{S|_{E^\perp} : S \in \mathcal{E}\}.$$

Now combine these two bases and the desired claim holds. $\square$

[1] In the finite-dimensional case, one can prove that an operator $T\colon V \to V$ is normal $\iff$ there is a polynomial p such that $T^* = p(T)$. Fugelde's theorem is immediate after this fact.

Since T_α is a self-adjoint operator of the finite dimensional space $S_k(\Gamma(1))$, there is a basis of $S_k(\Gamma(1))$ consisting of **Hecke eigenforms** which are an eigenfunctions for all Hecke operators.

Let

$$T_n := \sum_{\substack{d_1, d_2 \in \mathbb{Z}_{>0} \\ d_1 d_2 = n \\ d_2 | d_1}} T_{\mathrm{diag}(d_1, d_2)}.$$

Another way to think of this is that if Δ_n is the set of determinant n matrices in $\mathrm{Mat}_2(\mathbb{Z})$, then for each $\alpha \in \Delta_n$, we have

$$\Gamma(1)\alpha\Gamma(1) = \Gamma(1)\begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}\Gamma(1), \qquad d_1, d_2 \in \mathbb{Q}^+, d_1/d_2 \in \mathbb{Z}, d_1 d_2 = n.$$

Note that in the proof of Proposition 2.3, we may assume $N = 1$, so we see that $d_1, d_2 \in \mathbb{Z}_{>0}$. Therefore,

$$\Delta_n = \bigsqcup_{\substack{d_1, d_2 \in \mathbb{Z}_{>0} \\ d_1 d_2 = n \\ d_2 | d_1}} \Gamma(1) \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \Gamma(1).$$

If we write

$$\Delta_n = \bigsqcup_j \Gamma(1) \delta_{n,j},$$

then

$$T_n f = \sum_{\substack{d_1, d_2 \in \mathbb{Z}_{>0} \\ d_1 d_2 = n \\ d_2 | d_1}} f | T_{\mathrm{diag}(d_1, d_2)} = \sum_j f | \delta_{n,j}.$$

We now derive what these $\delta_{n,j}$ look like.

Lemma 2.9

$$\Delta_n = \bigsqcup_{\substack{a, d > 0 : ad = n \\ b \pmod d}} \Gamma(1) \begin{bmatrix} a & b \\ & d \end{bmatrix}.$$

Proof. Given any $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \Gamma(1) \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \Gamma(1)$. We claim we can perform elementary row operations to get the bottom left entry to zero. If $x = 0$, then swapping the first and second rows (and negating the top row afterwards) by multiplication by $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ yields the bottom left entry as zero. The idea now is that with elementary row operations, we can perform the Euclidean algorithm on (x, z) to get one of the left-hand entries equal to zero. Then after possibly swapping, the bottom left entry is zero.

Suppose we end up with $\begin{bmatrix} u & v \\ & w \end{bmatrix}$. Then by taking determinants, $uw = n$, and taking $R_1 - kR_2$, where $k \in \mathbb{Z}$ is such that $v - kw \in \{0, \ldots, w-1\}$, gives us the desired form.

It now suffices to show that this is a disjoint union. If there were a matrix $\begin{bmatrix} u & v \\ w & x \end{bmatrix} \in \Gamma(1)$ such that

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ & d_1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ & d_2 \end{bmatrix},$$

then

$$\begin{bmatrix} a_1 u & b_1 u + d_1 v \\ a_1 w & b_1 w + d_1 x \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ & d_2 \end{bmatrix}.$$

This implies $w = 0$, which combined with the fact that $ux - vw = 1$ implies that

$u, x \in \{\pm 1\}$, so $d_1 = \pm d_2$ and $a_1 = \pm a_2$ (where the signs are the same). After possibly multiplying by $-I$ on the left, we may assume $u = x = 1$ and $d_1 = d_2$. But this implies the top right entries of the matrix modulo $d_1 = d_2$ are equal, that is, $b_1 \equiv b_2 \pmod{d_1}$. □

Example 2.1 – The case for $n = 2$ may be enlightening. Any matrix with determinant 2 represents the same $\Gamma(1)$ left coset as an upper triangular matrix of the form $\begin{bmatrix} 2 & * \\ & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & * \\ & 2 \end{bmatrix}$.[1] In the former case, we can do row operations to make $* = 0$, hence making it diagonal. In the latter case, we can reduce the upper-right entry to either 0 or 1. Therefore,

$$\Delta_2 = \Gamma(1) \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \sqcup \Gamma(1) \begin{bmatrix} 1 & 0 \\ & 2 \end{bmatrix} \sqcup \Gamma(1) \begin{bmatrix} 1 & 1 \\ & 2 \end{bmatrix}.$$

[1] Indeed, one can perform row operations to make one of the entries in the first column zero. If the top entry is zero, then we may left multiply by $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ to get an upper triangular matrix.

Therefore,

$$T_n f(z) = \sum_{\substack{a,d>0 \\ ad=n}} \sum_{b \bmod d} f \bigg| \begin{bmatrix} a & b \\ & d \end{bmatrix} (z).$$

Proposition 2.10

If $f \in S_k(\Gamma(1))$ has Fourier expansion $\sum A(m) q^m$, then $T_n f$ has Fourier expansion $\sum B(m) q^m$, where

$$B(m) = \sum_{\substack{ad=n \\ a|m}} \left(\frac{a}{d}\right)^{k/2} d\, A\left(\frac{md}{a}\right).$$

Proof. Explicitly, we have

$$
\begin{aligned}
T_n f(z) &= \sum_{\substack{a,d>0 \\ ad=n}} \sum_{b \bmod d} f \Big| \begin{bmatrix} a & b \\ & d \end{bmatrix} (z) \\
&= \sum_{\substack{a,d>0 \\ ad=n}} \sum_{b \bmod d} n^{k/2} d^{-k} f\left(\frac{az+b}{d}\right) \\
&= \sum_{\substack{a,d>0 \\ ad=n}} \sum_{b \bmod d} \left(\frac{a}{d}\right)^{k/2} f\left(\frac{az+b}{d}\right) \\
&= \sum_{\substack{a,d>0 \\ ad=n}} \sum_{b \bmod d} \left(\frac{a}{d}\right)^{k/2} \sum_{m=1}^{\infty} A(m) e\left(\frac{amz}{d}\right) e\left(\frac{bm}{d}\right) \\
&= \sum_{m=1}^{\infty} \sum_{\substack{a,d>0 \\ ad=n}} \left(\frac{a}{d}\right)^{k/2} A(m) e\left(\frac{amz}{d}\right) \left(\sum_{b \bmod d} e\left(\frac{bm}{d}\right) \right).
\end{aligned}
$$

The blue sum is d if $d \mid m$, and zero otherwise. Therefore,

$$
T_n f(z) = \sum_{m=1}^{\infty} \sum_{\substack{a,d>0:ad=n \\ d \mid m}} d \left(\frac{a}{d}\right)^{k/2} e\left(\frac{amz}{d}\right) A(m)
$$

Since $B(m)$ is the coefficient associated to the $q^m = e(mz)$ term, we get

$$
B(m) = \sum_{\substack{ad=n \\ a \mid m}} \left(\frac{a}{d}\right)^{k/2} d\, A\left(\frac{md}{a}\right). \qquad \square
$$

Suppose f is a Hecke eigenform. We normalize the eigenvalues of the Hecke operators so that

$$
T_n f = n^{1-k/2} \lambda(n) f. \tag{2.1}
$$

Proposition 2.11

Let f be a Hecke eigenform together with the function $\lambda(n)$ so that (2.1) holds. Then

1. $A(1) \neq 0$.
2. If $A(1) = 1$, then $\lambda(n) = A(n)$ for all n.
3. If $A(1) = 1$, then A is multiplicative.

Proof. (1) We have

$$n^{1-k/2}\lambda(n)A(m) = \sum_{\substack{ad=n \\ a|m}} \left(\frac{a}{d}\right)^{k/2} d\, A\left(\frac{md}{a}\right).$$

If $(m, n) = 1$, then $a = 1$, $d = n$ is the only term in the RHS sum, so

$$\lambda(n)A(m) = A(nm).$$

If $m = 1$, then $\lambda(n)A(1) = A(n)$, which means $A(1) \neq 0$, otherwise $A(n) = 0$ for all n.

(2) Immediate from $\lambda(n)A(1) = A(n)$.

(3) Immediate from $\lambda(n)A(m) = A(nm)$ when $(m, n) = 1$. □

By part (1) in Proposition 2.11, we may assume $A(1) = 1$. We will say such a Hecke eigenform is *normalized*, and therefore there exists a normalized basis of Hecke eigenforms for $S_k(\Gamma(1))$.

Theorem 2.12
If f is a normalized Hecke eigenform, then

$$L(s, f) = \sum_n A(n)n^{-s} = \prod_p \left(1 - A(p)p^{-s} + p^{k-1-2s}\right)^{-1}.$$

Proof. Since A is multiplicative,

$$L(s, f) = \prod_p \left(\sum_{r=0}^{\infty} A(p^r)p^{-rs}\right).$$

For $r \in \mathbb{Z}_{\geq 0}$ and $k \geq 1$,

$$\begin{aligned} p^{1-k/2}\lambda(p)A(p^r) &= \sum_{\substack{ad=p \\ a|p^r}} \left(\frac{a}{d}\right)^{k/2} d\, A\left(\frac{p^r d}{a}\right) \\ &= \left(\frac{1}{p}\right)^{k/2} pA(p^{r+1}) + p^{k/2}A(p^{r-1}). \end{aligned}$$

Dividing both sides by $p^{1-k/2}$ and noting that $\lambda(p) = A(p)$, we have that

$$A(p^{r+1}) - A(p)A(p^r) + p^{k-1}A(p^{r-1}) = 0.$$

Working with formal power series in $\mathbb{C}[\![X]\!]$, and letting $A(x) = 0$ if $x \in \mathbb{Q} - \mathbb{Z}$, we have that

$$\begin{aligned}\left(1 - A(p)X + p^{k-1}X^2\right)\left(\sum_{r=0}^{\infty} A(p^r)X^r\right) \\ &= \sum_{r=0}^{\infty}\left(A(p^r) - A(p)A(p^{r-1}) + p^{k-1}A(p^{r-2})\right)X^r \\ &= A(1) + (A(p) - A(p)A(1))\,X \\ &= 1.\end{aligned}$$

Taking $X = p^{-s}$, we obtain the desired Euler product. □

3. An explicit example with the Ramanujan Tau function

We explicitly apply Hecke operators to Δ as an example. We have that

$$\Delta = q\prod_{n\geq 1}(1-q^n)^{24} \in S_{12}(\Gamma(1)),$$

which is a 1-dimensional vector space. Therefore, Δ is a Hecke eigenform. We have that $T_2\Delta = \Delta\Big|\begin{bmatrix}2 & \\ & 1\end{bmatrix} + \Delta\Big|\begin{bmatrix}1 & 0\\ & 2\end{bmatrix} + \Delta\Big|\begin{bmatrix}1 & 1\\ & 2\end{bmatrix}$. Expanding yields

$$\begin{aligned}\Delta\Big|\begin{bmatrix}2 & \\ & 1\end{bmatrix}(z) &= (2)^{12/2}\Delta\left(2z\right) = 2^6\Delta(q^2)\\ \Delta\Big|\begin{bmatrix}1 & 0\\ & 2\end{bmatrix}(z) &= (2)^{12/2}(2)^{-12}\Delta\left(\frac{z}{2}\right) = 2^{-6}\Delta(\sqrt{q})\\ \Delta\Big|\begin{bmatrix}1 & 1\\ & 2\end{bmatrix}(z) &= (2)^{12/2}(2)^{-12}\Delta\left(\frac{z+1}{2}\right) = 2^{-6}\Delta\left(-\sqrt{q}\right).\end{aligned}$$

Therefore, using the generatingfunctionology terminology where $f[q^n]$ is the coefficient of q^n in the q-expansion of a modular form f (e.g., $\tau(n) = \Delta[q^n]$),

$$T_2\Delta[q^n] = \begin{cases}2^6\tau(n/2) + 2^{-5}\tau(2n) & \text{if } n \text{ is even,}\\ 2^{-5}\tau(2n) & \text{if } n \text{ is odd.}\end{cases}$$

First, we have that

$$T_2\Delta[q] = 2^{-5}\tau(2) = -\frac{3}{4},$$

which proves that $T_2\Delta[q] \neq 0$. We know that Δ is an eigenform for T_2, so if we want

$$T_2\Delta = 2^{1-12/2}\lambda(2)\Delta = 2^{-5}\lambda(2)\Delta,$$

we must have

$$T_2\Delta[q] = 2^{-5}\lambda(2)\tau(1),$$

which implies

$$\lambda(2) = -24 = \tau(2).$$

We can check now that $\tau(2)\tau(3) = \tau(6)$. Indeed,

$$T_2\Delta[q^3] = 2^{-5}\lambda(2)\tau(3) = 2^{-5}\tau(6) \implies \tau(2)\tau(3) = \tau(6).$$

Since $\tau(1) = 1$, it is normalized. This immediately gives $\tau(n)\tau(m) = \tau(nm)$ if $(m,n) = 1$ by Proposition 2.11 (3). Theorem 2.12 gives us that

$$L(s,\Delta) = \sum_n \tau(n)n^{-s} = \prod_p \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1},$$

completing the proof.

References

[Axl97] Sheldon Jay Axler. *Linear algebra done right*, volume 2. springer New York, 1997.

[Bum97] Daniel Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.